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SIMULATION OF THE FLOW THROUGH POROUS LAYERS COMPOSED OF CONVERGING-DIVERGING CAPILLARY FISSURES OR TUBES

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In this paper, a porous medium is modelled by a network of converging-diverging capillaries which may be considered as fissures or tubes. This model makes it necessary to consider flows through capillary fissures or tubes. Therefore an analytical method for deriving the relationships between pressure drops, volumetric flow rates and velocities for the following fluids: Newtonian, polar, power-law, pseudoplastic (DeHaven and Sisko types) and Shulmanian, was developed. Next, considerations on the models of pore network for Newtonian and non-Newtonian fluids were presented. The models, similar to the schemes of central finite differences may provide a good basis for transforming the governing equations of a flow through the porous medium into a set of linear or quasi-linear algebraic equations. It was shown that the some coefficients in these algebraic equations depend on the kind of the capillary convergence.

Key words: laminar flow, symmetrically corrugated capillaries, capillary fissures or tubes, porous layer.

1. Introduction

Flows in porous media can be found in a number technological, medical and industrial applications (Bird *et al.* [1-3]). Fundamental and applied research on flow, heat and mass transfer in porous media has received increased attention during the past several decades due to the importance of these research areas in many engineering and biological applications. They can be modelled or approximated as transport phenomena through porous media and can be used in drying technology, thermal insulation, tissue replacement production, packed bed heat exchangers, geothermal systems, catalytic and biological reactors, gas and oil industries, etc

Fluid flows and transport phenomena through the classical “ground” or “soil” (Darcy, [4]) are encountered literally everywhere in everyday life, in nature (ground water), industries (composite materials, building materials, etc.) as well as in biosystems (aquifer ecosystems, human organs, etc.) and other domains such as e.g., membranes used in biofuel cell applications.

The reason is that except metals, some plastics and dense rocks, almost all solid and semisolid materials can be considered as “porous” in varying degrees. Hence, there exist many types of different technologies that depend on theories used to describe transport phenomena in porous media.

For example, it has been long discovered that sintering of granular materials (Chen *et al*. [5]) is not only a very large tonnage technology, where porous structures are significant, but it also finds applications in manufacturing ceramic products, papers and textiles.

There are many practical applications that can be modelled or approximated as transport through porous media. These applications have been discussed by Bear [6], Greenkorn [7], Nield and Bejan [8], Vafai [9-12], Hadim and Vafai [13], Vafai and Hadim [14].

In the works cited above the porous medium is viewed as a continuum with solid and fluid phases in thermal equilibrium, isotropic, homogeneous and saturated with an incompressible Newtonian fluid. Vafai and Tien [15] presented a comprehensive analysis of the generalized transport through porous media and developed a set of governing equations utilizing the local VAT (volume averaging theory/technique) or/and the REV (representative elementary volume) technique. The final forms of these equations can be found in the works by Amiri and Vafai [16], Alazmi and Vafai [17], Khanafer *et al*. [18]. Peng and Wu [19] describe a series of different experimental observations and associated theoretical investigations conducted to understand the transport phenomena with or without phase change and chemical reaction and concerning a wide range of practical applications. Fault and fracture zones are often highly-complex heterogeneities that can have a significant effect on the fluid flow within petroleum reservoirs on length scales of less than *1 μm* to more than *10 km*. It is therefore important to incorporate their properties in production simulation models. Harris *et al.* [20] describe some of the numerical techniques being used to model the effects of faults and fractures on fluid flow. Other theoretical models are groundwater models (Karamouz *et al.* [21], Yeh [22]) which have been used extensively for groundwater flow analysis, pollution transport and groundwater management.

Another way to study the flows in porous media is to use conceptual models; a great example of such models are PNMs (pore network models). These models have gained a lot of popularity among researchers since they are much more systematic than the real pore space of a soil and have been used in a variety of fields such as petroleum engineering, hydrology and soil physics. In these models, the soil pore space is modelled by a discrete network of pores that are connected by throats (Jivkov *et al.* [23]). Throats in PNMs may be prismatic or non-prismatic, mainly converging-diverging types (Xiong *et al.* [24]). The studies of the Newtonian flow in circular prismatic tubes (otherwise speaking: circular tubes of constant cross-sections) were performed by Mazaheri *et al.* [25], Joekar-Niaser *et al.* [26] and Nsir and Schafer [27]. The flow in non-prismatic tubes, namely in conical tubes was studied by Held and Celia [28], Hilpert *et al.* [29] and by Acharya *et al.* [30].

It has been found that at the bottom of rivers, lakes, seas and oceans an enhanced transport of solutes and particulate matter can be encountered in a thin layer, which comprises a tiny portion of the seawater layer from top and a tiny portion of the porewater layer from below, called a benthic layer. In this layer there may exist an interaction between the fluid flow and living media as in bioreactors (Chen [31]). The physicochemical and biological processes ongoing in the benthic layer cause that the fluid flowing through this layer behaves as a non-Newtonian fluid.

Flows of non-Newtonian fluids through porous media are frequently encountered in the petroleum industry (Vossoughi [32], Pearson and Tardy [33], Perrin *et al.* [34]). In exploitation of oil beds, an injection of polymer solutions into oil reservoirs is frequently applied to enhance oil recovery. Sometimes to achieve this aim, suspensions (frequently called slurries) of oil, coal and water are used (Vossoughi and Al-Husaini [35]).

The aim of this paper is to present the flows of the following fluids: Newtonian, polar, power-law, pseudoplastic and Shulmanian in convergent-divergent capillary fissures and tubes. As a result one obtains the formulae for the pressure drops, volumetric flow rates and inlet velocities for the flows through corrugated capillaries and the formulae for the velocities in the flows through a thin porous layer. Some remarks on pore network models are also presented.

2. Capillary description

Let us consider the fluid flow through convergent-divergent capillary fissures or tubes shown in Fig.1.

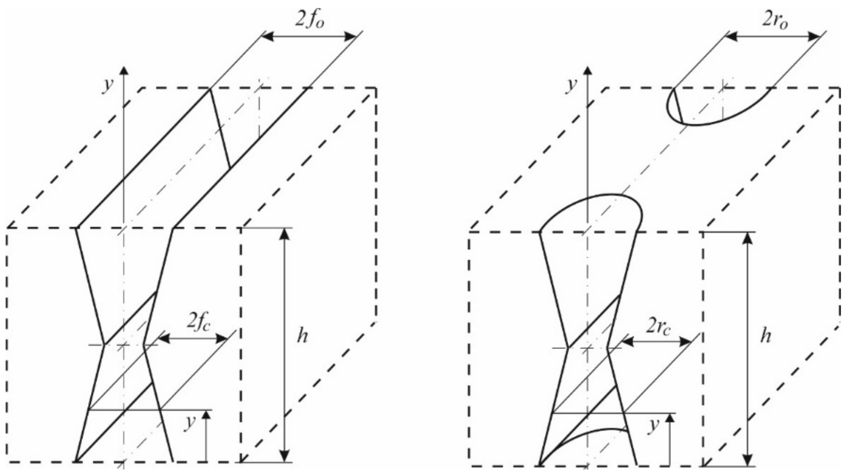

Fig.1. Schemes of the capillary fissures or tubes.

A cross dimension variation along the axis of the capillary is assumed to follow a simple power function in the local coordinate system, which may be written as (Acharya *et al*. [30])

$$\left.\begin{matrix} f_c(y) \\ r_c(y) \end{matrix}\right\} = \begin{matrix} D_o\left(1-\dfrac{y}{h}\right)^{n_c} & \text{for} & 0 \le y \le \dfrac{h}{2} \\ D_o\left(\dfrac{y}{h}\right)^{n_c} & & \dfrac{h}{2} \le y \le h \end{matrix} \tag{2.1}$$

where

$$D_o = \begin{cases} f_o & \text{for} \quad \text{capillary fissure} \\ r_o & \quad\quad\;\; \text{capillary tube} \end{cases} . \tag{2.2}$$

The shapes of the capillaries for different values of n_c are presented in Fig.2.

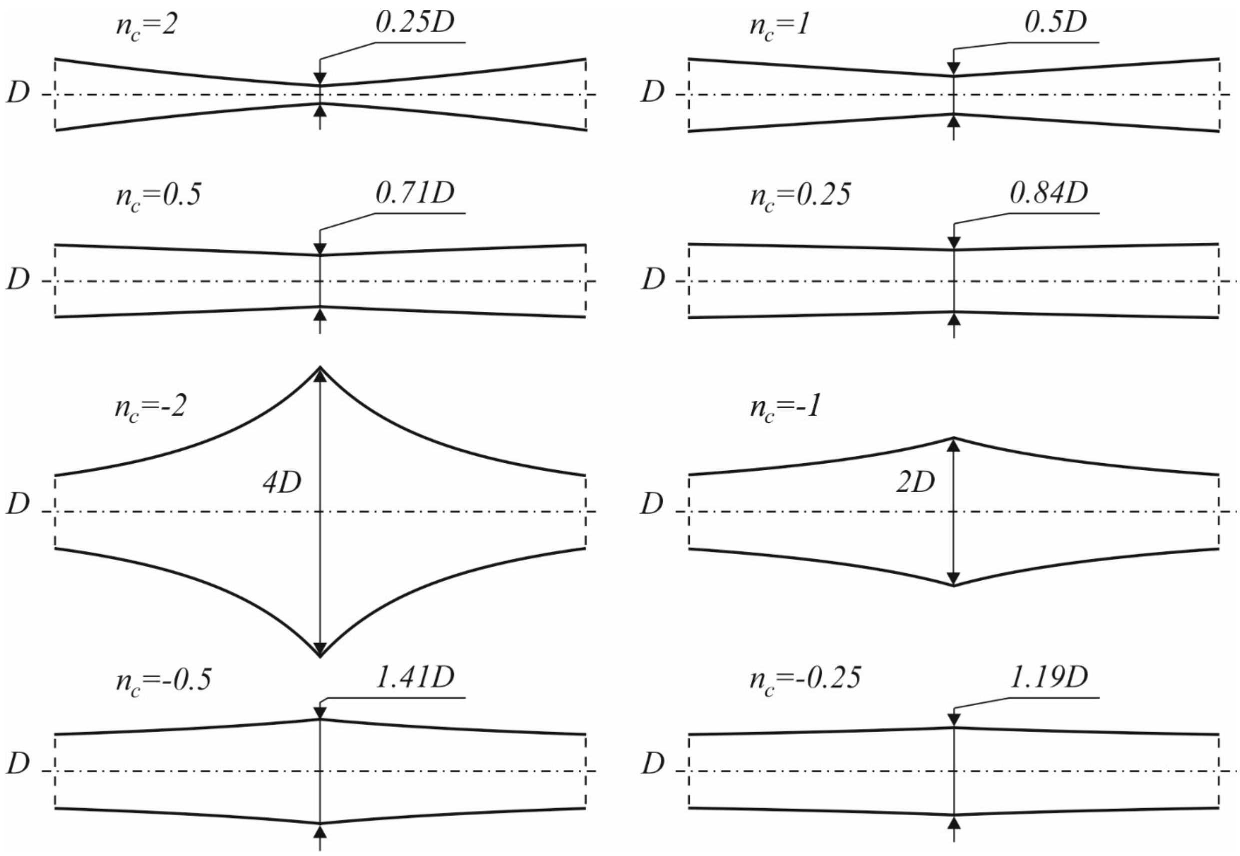

Fig.2. Capillary shapes for different values of n_c ; $D=2D_o$.

3. Flow of Newtonian fluids

Let us consider the flow of Newtonian fluids through capillaries of constant cross-sections. The velocities are given by the following expressions [36]:

– for the capillary fissure

$$\upsilon_f = \frac{f_c^2}{3\mu}\left(-\frac{dp}{dy}\right), \tag{3.1}$$

– for the capillary tube

$$\upsilon_t = \frac{r_c^2}{8\mu}\left(-\frac{dp}{dy}\right), \tag{3.2}$$

The volumetric flow rate Q is equal, respectively, to:

– for the capillary fissure

$$Q_f = 2f_c\upsilon_f = \frac{2f_c^3}{3\mu}\left(-\frac{dp}{dy}\right), \tag{3.3}$$

here Q_f is counted on the unit of a fissure width;

– for the capillary tube

$$Q_t = \pi r_c^2\upsilon_t = \frac{\pi r_c^4}{8\mu}\left(-\frac{dp}{dy}\right). \tag{3.4}$$

To find the pressure distribution in the capillary with a variable cross-section, we have the following expressions:
for the capillary fissure

$$\Delta p_f = 3\mu Q_f \int_0^{h/2} \frac{dy}{f_c^3}, \tag{3.5}$$

and for the capillary tube

$$\Delta p_t = \frac{16\mu Q_t}{\pi}\int_0^{h/2} \frac{dy}{r_c^4}. \tag{3.6}$$

Introducing expressions (2.1) and (2.2) into Eqs (3.5) and(3.6), we have

$$\Delta p_f = \frac{3\mu Q_f}{f_o^3}\int_0^{h/2} \frac{dy}{\left(1-\frac{y}{h}\right)^{3n_c}}$$

and after integration

$$\Delta p_f = \frac{3\mu h Q_f}{f_o^3}\left[\frac{2^{3n_c-1}-1}{3n_c-1}\right], \tag{3.7}$$

hence

$$Q_f = \frac{f_o^3}{3\mu}\left[\frac{3n_c - 1}{2^{3n_c - 1} - 1}\right]\left(\frac{\Delta p}{h}\right), \tag{3.8}$$

or

$$\Delta p_t = \frac{16\mu Q_t}{\pi r_o^4}\int_0^{h/2}\frac{dy}{\left(1 - \frac{y}{h}\right)^{4n_c}}$$

and

$$\Delta p_t = \frac{16\mu h Q_t}{\pi r_o^4}\left[\frac{2^{4n_c - 1} - 1}{4n_c - 1}\right], \tag{3.9}$$

hence

$$Q_t = \frac{\pi r_o^4}{16\mu}\left[\frac{4n_c - 1}{2^{4n_c - 1} - 1}\right]\left(\frac{\Delta p}{h}\right). \tag{3.10}$$

In the case of a thin porous layer, the inlet flow velocities through this layer are defined as follows

$$\upsilon_f = \frac{Q_f}{2f_o} = \frac{f_o^2}{6\mu}\left[\frac{3n_c - 1}{2^{3n_c - 1} - 1}\right]\left(-\frac{dp}{dy}\right) \tag{3.11}$$

and

$$\upsilon_t = \frac{Q_t}{\pi r_o^2} = \frac{r_o^2}{16\mu}\left[\frac{4n_c - 1}{2^{4n_c - 1} - 1}\right]\left(-\frac{dp}{dy}\right). \tag{3.12}$$

Note that if $n_c = 0$, these formulae become the same as formulae (3.1) and (3.2).

4. Flows of polar fluids

Some flows of slurries can be modelled by the flows of polar fluids. This modelling is correct for the slurries being homogeneous suspensions. Two polar fluids are frequently used, namely: a micropolar fluid and couple-stress fluid.

4.1. Flow of micropolar fluids

For the flow of a micropolar fluid in capillaries there are the following expressions for the velocities [36]: for the capillary fissure

$$\upsilon_f = \frac{f_c^2}{3\mu}\frac{\mu}{\mu + k}\left(-\frac{dp}{dy}\right) \tag{4.1}$$

and for the capillary tube

$$\upsilon_t = \frac{r_c^2}{8\mu}\frac{\mu}{\mu+k}\left(-\frac{dp}{dy}\right) \tag{4.2}$$

where k is the vortex (coupling) viscosity; if $k=0$, then the flow becomes Newtonian and Eqs (3.1) and (3.2) are important. It results from the fact that multiplying, respectively, all formulae (3.7)-(3.10) by the coefficient $\frac{\mu}{\mu+k}$, we will obtain the formulae adequate to the flow of a micropolar fluid in capillaries.

4.2. Flow of couple-stress fluids

Let us consider the flow of a couple-stress fluid in capillaries of constant cross-sections. The flow velocities are given by the following expressions [36]:

- for the capillary fissure

$$\upsilon_f = \frac{f_c^2}{3\mu}\left(1-\frac{3l^2}{f_c^2}\right)\left(-\frac{dp}{dy}\right) \tag{4.3}$$

and

- for the capillary tube

$$\upsilon_t = \frac{r_e^2}{8\mu}\left(1-\frac{4l^2}{r_c^2}\right)\left(-\frac{dp}{dy}\right) \tag{4.4}$$

where $l^2 = \frac{\eta}{\mu}$ and η is a material coefficient associated with couple-stresses.

The flow rates are equal to, respectively

$$Q_f = \frac{2f_c^3}{3\mu}\left(1-\frac{3l^2}{f_c^2}\right)\left(-\frac{dp}{dy}\right), \tag{4.5}$$

$$Q_t = \frac{\pi r_c^4}{8\mu}\left(1-\frac{4l^2}{r_c^2}\right)\left(-\frac{dp}{dy}\right). \tag{4.6}$$

To determine the flow in the capillaries of variable cross-sections let us assume the following principle of superposition that the total flow rate Q_c is equal to

$$Q_c = Q_N + Q_A \tag{4.7}$$

where Q_c is either Q_f or Q_t, Q_N is a respective Newtonian flow rate and Q_A is a respective additional flow rate.

Thus, we have, respectively

$$Q_f = Q_{fN} + Q_{fA} \tag{4.8}$$

where

$$Q_{fN} = \frac{2f_c^3}{3\mu}\left(-\frac{dp}{dy}\right), \tag{4.9}$$

$$Q_{fA} = -\frac{2f_c l^2}{\mu}\left(-\frac{dp}{dy}\right) \tag{4.10}$$

and

$$Q_t = Q_{tN} + Q_{tA} \tag{4.11}$$

$$Q_{tN} = \frac{\pi r_c^4}{8\mu}\left(-\frac{dp}{dy}\right), \tag{4.12}$$

$$Q_{tA} = -\frac{\pi r_c^2 l^2}{2\mu}\left(-\frac{dp}{dy}\right). \tag{4.13}$$

Proceeding similarly as for Newtonian fluids and taking into account an assumption that the flow rate through capillaries of variable cross-sections must be always the same, we have subsequent results:

– for the capillary fissure

$$Q_{fN} = \frac{f_o^3}{3\mu}\left[\frac{3n_c - 1}{2^{3n_c - 1} - 1}\right]\left(\frac{\Delta p_f}{h}\right), \tag{4.14}$$

$$Q_{fA} = -\frac{l^2 f_o}{\mu}\left[\frac{n_c - 1}{2^{n_c - 1} - 1}\right]\left(\frac{\Delta p_f}{h}\right) \tag{4.15}$$

and

$$Q_f = \frac{f_o^3}{3\mu}\left[\frac{3n_c - 1}{2^{3n_c - 1} - 1} - \frac{3l^2}{f_o^2}\frac{n_c - 1}{2^{n_c - 1} - 1}\right]\left(\frac{\Delta p_f}{h}\right), \tag{4.16}$$

$$\upsilon_f = \frac{f_o^2}{6\mu}\left[\frac{3n_c - 1}{2^{3n_c - 1} - 1} - \frac{3l^2}{f_o^2}\frac{n_c - 1}{2^{n_c - 1} - 1}\right]\left(-\frac{dp}{dy}\right); \tag{4.17}$$

– for the capillary tube

$$Q_{tN} = \frac{\pi r_o^4}{16\mu}\left[\frac{4n_c - 1}{2^{4n_c - 1} - 1}\right]\left(\frac{\Delta p_t}{h}\right), \tag{4.18}$$

$$Q_{tA} = -\frac{\pi l^2 r_o^2}{4\mu}\left[\frac{2n_c - 1}{2^{2n_c - 1} - 1}\right]\left(\frac{\Delta p_t}{h}\right) \tag{4.19}$$

and

$$Q_t = \frac{\pi r_o^4}{16\mu}\left[\frac{4n_c - 1}{2^{4n_c - 1} - 1} - \frac{4l^2}{r_o^2}\frac{2n_c - 1}{2^{2n_c - 1} - 1}\right]\left(\frac{\Delta p_t}{h}\right), \tag{4.20}$$

$$\upsilon_t = \frac{r_o^2}{16\mu}\left[\frac{4n_c - 1}{2^{4n_c - 1} - 1} - \frac{4l^2}{r_o^2}\frac{2n_c - 1}{2^{2n_c - 1} - 1}\right]\left(-\frac{dp}{dy}\right). \tag{4.21}$$

Note that for the couple-stress fluid being in a maceration state the Darcy law is frequently expressed as follows [37]

$$\mathbf{v} = \frac{\Phi_n}{\mu(1-\beta)}(-\nabla p) \tag{4.22}$$

where $\mathbf{v}$ is the velocity vector, ∇p is the pressure gradient, μ is the shear viscosity, Φ_n is the permeability of the porous medium, and

$$\beta = \frac{\eta}{\mu\Phi_n}, \tag{4.23}$$

here η is an already known material constant.
The permeability of the porous medium can be expressed as

$$\Phi_n = \frac{f_c^2 \varphi_p}{3} \tag{4.24}$$

for capillary fissures and

$$\Phi_n = \frac{r_c^2 \varphi_p}{4} \tag{4.25}$$

for capillary tubes coefficient φ_p is the porosity of the porous medium; for a single capillary $\varphi_p = 1$. Taking into account these considerations we can record that the formulae (4.3) and (4.4) express the Darcy law for a couple-stress fluid in a one-dimensional form.

5. Flows of power-law fluids

Let us consider the flow of power-law fluids (Ostwald-de Waele fluids [38, 39]) through capillaries of constant cross-sections. The velocities are given by the following formulae [36]:

– in the capillary fissure

$$\upsilon_f = \frac{f_c^{m+1}}{(m+2)\mu}\left(-\frac{dp}{dy}\right)^m , \tag{5.1}$$

– in the capillary tube

$$\upsilon_t = \frac{r_c^{m+1}}{2^m (m+3)\mu}\left(-\frac{dp}{dy}\right)^m . \tag{5.2}$$

The volumetric flow rates Q_f, Q_t are equal to, respectively

$$Q_f = \frac{2 f_c^{m+2}}{(m+2)\mu}\left(-\frac{dp}{dy}\right)^m , \tag{5.3}$$

$$Q_t = \frac{\pi r_c^{m+3}}{2^m (m+3)\mu}\left(-\frac{dp}{dy}\right)^m . \tag{5.4}$$

To find the pressure distribution in capillaries with variable cross-sections we have the following expressions:

– for the capillary fissure

$$\Delta p_f = 2\left[\frac{\mu(m+2)Q_f}{2}\right]^{1/m} \int_0^{h/2} \frac{dy}{f_c^{\frac{m+2}{m}}} . \tag{5.5}$$

– for the capillary tube

$$\Delta p_t = 4\left[\frac{\mu(m+3)Q_t}{\pi}\right]^{1/m} \int_0^{h/2} \frac{dy}{r_c^{\frac{m+3}{m}}} . \tag{5.6}$$

Introducing here the expressions (2.1) and (2.2) we will obtain after integration:

– for the capillary fissure

$$\Delta p_f = 2\left[\frac{\mu(m+2)Q_f}{2}\right]^{1/m} \frac{h\left(2^{\alpha_f - 1} - 1\right)}{f_o^{\frac{m+2}{m}}\left(\alpha_f - 1\right)} , \tag{5.7}$$

$$Q_f = \frac{2 f_o^{m+2}}{(m+2)\mu}\left[\frac{\alpha_f - 1}{2\left(2^{\alpha_f - 1} - 1\right)}\right]^m \left(\frac{\Delta p_f}{h}\right)^m , \tag{5.8}$$

$$\upsilon_f = \frac{f_o^{m+1}}{(m+2)\mu}\left[\frac{\alpha_f - 1}{2\left(2^{\alpha_f - 1} - 1\right)}\right]^m \left(-\frac{dp}{dy}\right)^m ; \tag{5.9}$$

– for the capillary tube

$$\Delta p_t = 4\left[\frac{\mu(m+3)Q_t}{\pi}\right]^{1/m} \frac{h\left(2^{\alpha_t - 1} - 1\right)}{r_o^{\frac{m+3}{m}}\left(\alpha_t - 1\right)}, \tag{5.10}$$

$$Q_t = \frac{\pi r_o^{m+3}}{2^m (m+3)\mu}\left[\frac{\alpha_t - 1}{2\left(2^{\alpha_t - 1} - 1\right)}\right]^m \left(\frac{\Delta p_t}{h}\right)^m , \tag{5.11}$$

$$\upsilon_t = \frac{r_o^{m+1}}{2^m (m+3)\mu}\left[\frac{\alpha_t - 1}{2\left(2^{\alpha_t - 1} - 1\right)}\right]^m \left(-\frac{dp}{dy}\right)^m \tag{5.12}$$

where

$$\alpha_f = \frac{n_c(m+2)}{m}, \qquad \alpha_t = \frac{n_c(m+3)}{m}. \tag{5.13}$$

Note that these formulae for $m = 1$ reduce to the formulae adequate to the Newtonian fluids.

6. Flow of DeHaven type fluids

DeHaven fluids are pseudoplastic fluids which are characterized by a non-linear relationship between the shear stress and the shear strain rate; to be more precise it can be stated that the shear strain rate is a non-linear function of the shear stress [40, 41]. The flow velocities in capillaries are given by the following expressions [36]:
in the capillary fissure

$$\upsilon_f = \frac{f_c^2}{3\mu}\left[1 + \frac{3k_i f_c^{n_i}}{(n_i + 3)}\left(-\frac{dp}{dy}\right)^{n_i}\right]\left(-\frac{dp}{dy}\right), \tag{6.1}$$

in the capillary tube

$$\upsilon_t = \frac{r_c^2}{8\mu}\left[1 + \frac{4k_i r_c^{n_i}}{2^{n_i}(n_i + 4)}\left(-\frac{dp}{dy}\right)^{n_i}\right]\left(-\frac{dp}{dy}\right) \tag{6.2}$$

where k_i and n_i are material coefficients characteristic for a given model of the DeHaven fluid. Note that the following – well known – models of fluids may be reduced to the DeHaven model, namely: Meter, Ellis, Rotem-Shinnar, Ree-Eyring, Rabinowitsch, Reiner-Philipoff, Peek-McLean and Seely [36].
The flow rates are equal to, respectively

$$Q_f = \frac{2f_c^3}{3\mu}\left[1 + \frac{3k_i f_c^{n_i}}{(n_i+3)}\left(-\frac{dp}{dy}\right)^{n_i}\right]\left(-\frac{dp}{dy}\right), \tag{6.3}$$

$$Q_t = \frac{\pi r_c^4}{8\mu}\left[1 + \frac{4k_i r_c^{n_i}}{2^{n_i}(n_i+4)}\left(-\frac{dp}{dy}\right)^{n_i}\right]\left(-\frac{dp}{dy}\right). \tag{6.4}$$

Assuming the principle of superposition (as in Section 4), we may write

$$Q_c = Q_N + Q_P \tag{6.5}$$

where Q_c is either Q_f or Q_t, Q_N is a Newtonian flow rate, Q_P denotes a power-law flow rate.
Thus, we have, respectively

$$Q_f = Q_{fN} + Q_{fP} \tag{6.6}$$

where

$$Q_{fN} = \frac{2f_c^3}{3\mu}\left(-\frac{dp}{dy}\right), \tag{6.7}$$

$$Q_{fP} = \frac{2k_i f_c^{n_i+3}}{\mu(n_i+3)}\left(-\frac{dp}{dy}\right)^{n_i+1} \tag{6.8}$$

and

$$Q_t = Q_{tN} + Q_{tP} \tag{6.9}$$

$$Q_{tN} = \frac{\pi r_c^4}{8\mu}\left(-\frac{dp}{dy}\right), \tag{6.10}$$

$$Q_{tP} = \frac{\pi k_i r_c^{n_i+4}}{2^{n_i+1}\mu(n_i+4)}\left(-\frac{dp}{dy}\right)^{n_i+1}. \tag{6.11}$$

Taking into account the results presented in Sections 3 and 5 we may write:

- for the capillary fissure

$$Q_{fN} = \frac{f_o^3}{3\mu}\left[\frac{3n_c - 1}{2^{3n_c-1} - 1}\right]\left(\frac{\Delta p_f}{h}\right), \tag{6.12}$$

$$Q_{fP} = \frac{k_i f_o^{n_i+3}}{2^{n_i}\mu(n_i+3)}\left[\frac{\alpha_f - 1}{2^{\alpha_f-1} - 1}\right]^{n_i+1}\left(\frac{\Delta p_f}{h}\right)^{n_i+1}, \tag{6.13}$$

$$Q_f = \frac{f_o^3}{3\mu}\left[\left(\frac{3n_c - 1}{2^{3n_c-1} - 1}\right) + \frac{3k_i f_o^{n_i}}{2^{n_i}(n_i+3)}\left(\frac{\alpha_f - 1}{2^{\alpha_f-1} - 1}\right)^{n_i+1}\left(\frac{\Delta p_f}{h}\right)^{n_i}\right]\left(\frac{\Delta p_f}{h}\right), \tag{6.14}$$

$$\upsilon_f = \frac{f_o^2}{6\mu}\left[\left(\frac{3n_c - 1}{2^{3n_c-1} - 1}\right) + \frac{3k_i f_o^{n_i}}{2^{n_i}(n_i+3)}\left(\frac{\alpha_f - 1}{2^{\alpha_f-1} - 1}\right)^{n_i+1}\left(-\frac{dp}{dy}\right)^{n_i}\right]\left(-\frac{dp}{dy}\right) \tag{6.15}$$

where

$$\alpha_f = n_c\left(\frac{n_i+3}{n_i+1}\right); \tag{6.16}$$

– for the capillary tube

$$Q_{tN} = \frac{\pi r_o^4}{16\mu}\left[\frac{4n_c - 1}{2^{4n_c-1} - 1}\right]\left(\frac{\Delta p_t}{h}\right), \tag{6.17}$$

$$Q_{tP} = \frac{\pi k_i r_o^{n_i+4}}{4^{n_i+1}\mu(n_i+4)}\left[\frac{\alpha_t - 1}{2^{\alpha_t-1} - 1}\right]^{n_i+1}\left(\frac{\Delta p_t}{h}\right)^{n_i+1}, \tag{6.18}$$

$$Q_t = \frac{\pi r_o^4}{16\mu}\left[\frac{4n_c - 1}{2^{4n_c-1} - 1} + \frac{4k_i r_o^{n_i}}{4^{n_i}(n_i+4)}\left(\frac{\alpha_t - 1}{2^{\alpha_t-1} - 1}\right)^{n_i+1}\left(\frac{\Delta p_t}{h}\right)^{n_i}\right]\left(\frac{\Delta p_t}{h}\right), \tag{6.19}$$

$$\upsilon_t = \frac{r_o^2}{8\mu}\left[\frac{1}{2}\left(\frac{4n_c - 1}{2^{4n_c-1} - 1}\right) + \frac{4k_i r_o^{n_i}}{2^{n_i}(n_i+4)}\left(\frac{\alpha_t - 1}{2^{\alpha_t-1} - 1}\right)^{n_i+1}\left(-\frac{dp}{dy}\right)^{n_i}\right]\left(-\frac{dp}{dy}\right) \tag{6.20}$$

where

$$\alpha_t = n_c\left(\frac{n_i+4}{n_i+1}\right). \tag{6.21}$$

Note that these formulae for n_c reduce to the formulae adequate to the flow in capillaries of constant cross-sections.

7. Flow of Sisko type fluids

In this section, we will consider another group of pseudoplastic fluids which are also characterized by a nonlinear relationship between the shear stress and the shear strain rate; the shear stress is here a nonlinear function of the shear strain rate. One of more general models of this kind is a Sisko model [42, 43]. The flow velocities in capillaries are given by the following expressions [36]:
in the capillary fissure

$$\upsilon_f = \frac{f_c^2}{3\mu}\left[1-\frac{3\mu_i f_c^{n_i}}{\mu^{n_i+1}\left(n_i+3\right)}\left(-\frac{dp}{dy}\right)^{n_i}\right]\left(-\frac{dp}{dy}\right), \tag{7.1}$$

in the capillary tube

$$\upsilon_t = \frac{r_c^2}{8\mu}\left[1-\frac{4\mu_i r_c^{n_i}}{2^{n_i}\mu^{n_i+1}\left(n_i+4\right)}\left(-\frac{dp}{dy}\right)^{n_i}\right]\left(-\frac{dp}{dy}\right) \tag{7.2}$$

where μ_i and n_i are material coefficients characteristic for a given model of the Sisko fluid. Note that the following – well known – models of fluids may be reduced to the Sisko model, namely: Carreau-Yasuda, Elsharkawy-Hamrock, Prandtl, Eyring-Sutterby, Gecim-Winer and Bair-Winer [36].
The flow rates are given by, respectively

$$Q_f = \frac{2f_c^3}{3\mu}\left[1-\frac{3\mu_i f_c^{n_i}}{\mu^{n_i+1}\left(n_i+3\right)}\left(-\frac{dp}{dy}\right)^{n_i}\right]\left(-\frac{dp}{dy}\right), \tag{7.3}$$

$$Q_t = \frac{\pi r_c^4}{8\mu}\left[1-\frac{4\mu_i r_c^{n_i}}{2^{n_i}\mu^{n_i+1}\left(n_i+4\right)}\left(-\frac{dp}{dy}\right)^{n_i}\right]\left(-\frac{dp}{dy}\right). \tag{7.4}$$

Assuming the principle of superposition – as in the previous section – we may write

$$Q_c = Q_N - Q_P \tag{7.5}$$

where Q_c is either Q_f or Q_t, Q_N is a Newtonian flow rate, Q_P denotes a power-law flow rate.
Thus , we have, respectively

$$Q_f = Q_{fN} - Q_{fP} \tag{7.6}$$

where

$$Q_{fN} = \frac{2f_c^3}{3\mu}\left(-\frac{dp}{dy}\right), \tag{7.7}$$

$$Q_{fP}=\frac{2\mu_i f_c^{n_i+3}}{\mu^{n_i+2}\left(n_i+3\right)}\left(-\frac{dp}{dy}\right)^{n_i+1} \tag{7.8}$$

and

$$Q_t=Q_{tN}-Q_{tP} \tag{7.9}$$

where

$$Q_{tN}=\frac{\pi r_c^4}{8\mu}\left(-\frac{dp}{dy}\right), \tag{7.10}$$

$$Q_{tP}=\frac{\pi\mu_i r_c^{n_i+4}}{2^{n_i+1}\mu^{n_i+2}\left(n_i+4\right)}\left(-\frac{dp}{dy}\right)^{n_i+1}. \tag{7.11}$$

Taking into account the results presented in Sections 3 and 5, we may write:

- for the capillary fissure

$$Q_{fN}=\frac{f_o^3}{3\mu}\left[\frac{3n_c-1}{2^{3n_c-1}-1}\right]\left(\frac{\Delta p_f}{h}\right), \tag{7.12}$$

$$Q_{fP}=\frac{\mu_i f_o^{n_i+3}}{2^{n_i}\mu^{n_i+2}\left(n_i+3\right)}\left[\frac{\alpha_f-1}{2^{\alpha_f-1}-1}\right]^{n_i+1}\left(\frac{\Delta p_f}{h}\right)^{n_i+1}, \tag{7.13}$$

$$Q_f=\frac{f_o^3}{3\mu}\left[\left(\frac{3n_c-1}{2^{3n_c-1}-1}\right)-\frac{3\mu_i f_o^{n_i}}{2^{n_i}\mu^{n_i+1}\left(n_i+3\right)}\left(\frac{\alpha_f-1}{2^{\alpha_f-1}-1}\right)^{n_i+1}\left(\frac{\Delta p_f}{h}\right)^{n_i}\right]\left(\frac{\Delta p_f}{h}\right), \tag{7.14}$$

$$\upsilon_f=\frac{f_o^2}{6\mu}\left[\left(\frac{3n_c-1}{2^{3n_c-1}-1}\right)-\frac{3\mu_i f_o^{n_i}}{2^{n_i}\mu^{n_i+1}\left(n_i+3\right)}\left(\frac{\alpha_f-1}{2^{\alpha_f-1}-1}\right)^{n_i+1}\left(-\frac{dp}{dy}\right)^{n_i}\right]\left(-\frac{dp}{dy}\right); \tag{7.15}$$

where α_f is given by formula (6.16);

- for the capillary tube

$$Q_{tN}=\frac{\pi r_o^4}{16\mu}\left[\frac{4n_c-1}{2^{4n_c-1}-1}\right]\left(\frac{\Delta p_t}{h}\right), \tag{7.16}$$

$$Q_{tP}=\frac{\pi\mu_i r_o^{n_i+4}}{4^{n_i+1}\mu^{n_i+2}\left(n_i+4\right)}\left[\frac{\alpha_t-1}{2^{\alpha_t-1}-1}\right]^{n_i+1}\left(\frac{\Delta p_t}{h}\right)^{n_i+1}, \tag{7.17}$$

$$Q_t = \frac{\pi r_o^4}{16\mu}\left[\frac{4n_c - 1}{2^{4n_c - 1} - 1} - \frac{4\mu_i r_o^{n_i}}{4^{n_i}\mu^{n_i+1}(n_i+4)}\left(\frac{\alpha_t - 1}{2^{\alpha_t - 1} - 1}\right)^{n_i+1}\left(\frac{\Delta p_t}{h}\right)^{n_i}\right]\left(\frac{\Delta p_t}{h}\right), \tag{7.18}$$

$$\upsilon_t = \frac{r_o^2}{16\mu}\left[\frac{4n_c - 1}{2^{4n_c - 1} - 1} - \frac{16\mu_i r_o^{n_i}}{4^{n_i+1}\mu^{n_i+1}(n_i+4)}\left(\frac{\alpha_t - 1}{2^{\alpha_t - 1} - 1}\right)^{n_i+1}\left(-\frac{dp}{dy}\right)^{n_i}\right]\left(-\frac{dp}{dy}\right). \tag{7.19}$$

where α_t is given by formula (6.21).

8. Flow of Shulman type fluids

Many fluids of engineering interest appear to exhibit yield behaviour, where flow occurs only when the imposed stress exceeds a critical yield stress. To describe the rheological behaviour of such a viscoplastic fluid the non-linear model of Shulman [44, 45] may be used.

The flow velocities of the Shulman type fluids in capillaries of constant cross-sections are given by the expressions presented in Tabs 1 and 2 [36]

Table 1. Formulae for the velocity flow of the Shulman type fluids through capillary fissures.

Model of fluid	Velocity flow υ_f through capillary fissures
(8.1) Shulman	$\upsilon_f = \chi^{\frac{m+2n}{n}}\sum_{i=0}^{m}(-1)^i\frac{n}{m+2n-i}C_m^i\left(\chi^{-\frac{m+2n-i}{n}} - 1\right)\frac{b^{\frac{m+n}{n}}}{\mu}\left(-\frac{dp}{dy}\right)^{\frac{m}{n}}$
(8.2) Casson $m = n$	$\upsilon_f = \chi^3\sum_{i=0}^{n}(-1)^i\frac{n}{3n-i}C_n^i\left(\chi^{-\frac{3n-i}{n}} - 1\right)\frac{b^2}{\mu}\left(-\frac{dp}{dy}\right)$
(8.3) Casson "simple" $m = n = 2$	$\upsilon_f = \left(1 - \frac{12}{5}\chi^{\frac{1}{2}} + \frac{3}{2}\chi - \frac{\chi^3}{10}\right)\frac{b^2}{3\mu}\left(-\frac{dp}{dy}\right)$
(8.4) Vočadlo $m = 1$	$\upsilon_f = \left(1 - \frac{1+2n}{2n}\chi^{\frac{1}{n}} + \frac{1}{2n}\chi^{\frac{1+2n}{n}}\right)\frac{nb^{\frac{1+2}{n}}}{(1+2n)\mu}\left(-\frac{dp}{dy}\right)^{\frac{1}{n}}$
(8.5) Herschel-Bulkley $n = 1$	$\upsilon_f = (1-\chi)^{m+1}\left(1 + \frac{\chi}{m+1}\right)\frac{b^{m+1}}{(m+2)\mu}\left(-\frac{dp}{dy}\right)^m$
(8.6) Bingham $m = n = 1$	$\upsilon_f = \left(1 - \frac{3}{2}\chi + \frac{1}{2}\chi^3\right)\frac{b^2}{3\mu}\left(-\frac{dp}{dy}\right)$
(8.7) Ostwald-de Waele $n = 1, \tau_0 = 0$	$\upsilon_f = \frac{b^{m+1}}{(m+2)\mu}\left(-\frac{dp}{dy}\right)^m$

Table 2. Formulae for the velocity flow of the Shulman type fluids through capillary tubes.

Model of fluid	Velocity flow υ_t through capillary tubes
(8.8) Shulman	$\upsilon_t = \Upsilon^{\frac{m+3n}{n}} \sum_{i=0}^{m} (-1)^i \frac{n}{m+3n-i} C_m^i \left(\Upsilon^{-\frac{m+3n-i}{n}} - 1 \right) \frac{r_c^{\frac{m+n}{n}}}{2^{\frac{m}{n}} \mu} \left(-\frac{dp}{dy} \right)^{\frac{m}{n}}$
(8.9) Casson $m=n$	$\upsilon_t = \Upsilon^4 \sum_{i=0}^{n} (-1)^i \frac{n}{4n-i} C_n^i \left(\Upsilon^{-\frac{4n-i}{n}} - 1 \right) \frac{r_c^2}{2\mu} \left(-\frac{dp}{dy} \right)$
(8.10) Casson "simple" $m=n=2$	$\upsilon_t = \left(1 - \frac{16}{7} \Upsilon^{\frac{1}{2}} + \frac{4}{3} \Upsilon - \frac{\Upsilon^4}{21} \right) \frac{r_c^2}{8\mu} \left(-\frac{dp}{dy} \right)$
(8.11) Vočadlo $m=1$	$\upsilon_t = \left(1 - \frac{3n+1}{3n} \Upsilon^{\frac{1}{n}} + \frac{1}{3n} \Upsilon^{\frac{3n+1}{n}} \right) \frac{n r_c^{\frac{n+1}{n}}}{2^{\frac{1}{n}} (3n+1) \mu} \left(-\frac{dp}{dy} \right)^{\frac{1}{n}}$
(8.12) Herschel-Bulkley $n=1$	$\upsilon_t = (1-\Upsilon)^{m+1} \left[1 + \frac{2\Upsilon}{m+2} + \frac{2\Upsilon^2}{(m+1)(m+2)} \right] \frac{r_c^{m+1}}{2^m (m+3) \mu} \left(-\frac{dp}{dy} \right)^m$
(8.13) Bingham $m=n=1$	$\upsilon_t = \left(1 - \frac{4}{3} \Upsilon + \frac{1}{3} \Upsilon^4 \right) \frac{r_c^2}{8\mu} \left(-\frac{dp}{dy} \right)$
(8.14) Ostwald-de Waele $n=1$, $\tau_0 = 0$	$\upsilon_t = \frac{r_c^{m+1}}{2^m (m+3) \mu} \left(-\frac{dp}{dy} \right)^m$

Here χ and Υ denote; respectively

$$\chi = \frac{\tau_0}{\tau_w} = \frac{f_0}{f_c}, \qquad \Upsilon = \frac{\tau_0}{\tau_w} = \frac{r_0}{r_c} \tag{8.15}$$

whereas

$$C_m^i = \frac{m!}{i!(m-i)!};$$

and: τ_0 is the yield shear stress, τ_w is the wall shear stress, f_0 is a half thickness of the core flow in the capillary fissure of thickness equals to $2f_c$, r_0 is the radius of the core flow in the capillary tube of radius r_c. Assuming the affinity of the velocity field it can be accepted that the values of χ and Υ will be constant in the flows through capillaries of variable cross-sections.
The volumetric flow rates are equal, respectively

$$Q_f = 2 f_c \upsilon_f \quad \text{or} \quad Q_f = \pi r_c^2 \upsilon_t, \tag{8.16}$$

then we may write that the velocity flow for an arbitrary fluid model is equal to

$$\upsilon_f = F_{(n)}^{(m)} \frac{f_c^{\frac{m+n}{n}}}{\mu}\left(-\frac{dp}{dy}\right)^{\frac{m}{n}}, \tag{8.17}$$

$$\upsilon_t = T_{(n)}^{(m)} \frac{r_c^{\frac{m+n}{n}}}{2^{\frac{m}{n}}\mu}\left(-\frac{dp}{dy}\right)^{\frac{m}{n}} \tag{8.18}$$

and

$$Q_f = F_{(n)}^{(m)} \frac{2 f_c^{\frac{m+2n}{n}}}{\mu}\left(-\frac{dp}{dy}\right)^{\frac{m}{n}}, \tag{8.19}$$

$$Q_t = T_{(n)}^{(m)} \frac{\pi r_c^{\frac{m+3n}{n}}}{2^{\frac{m}{n}}\mu}\left(-\frac{dp}{dy}\right)^{\frac{m}{n}} \tag{8.20}$$

where $F_{(n)}^{(m)}$ and $T_{(n)}^{(m)}$ are constant quantities, which are, e.g., for the Shulman fluid and for the Ostwald-de Waele fluid, equal to, respectively

$$F_{(n)}^{(m)} = \chi^{\frac{m+2n}{n}} \sum_{i=0}^{m} (-1)^i \frac{n}{m+2n-i} C_m^i \left(\chi^{-\frac{m+2n-i}{n}} - 1\right),$$

$$T_{(n)}^{(m)} = \Upsilon^{\frac{m+3n}{n}} \sum_{i=0}^{m} (-1)^i \frac{n}{m+3n-i} C_m^i \left(\Upsilon^{-\frac{m+3n-i}{n}} - 1\right),$$

$$F_{(1)}^{(m)} = \frac{1}{m+2},$$

$$T_{(1)}^{(m)} = \frac{1}{m+3}.$$

Note that for flows with a large core when $\chi \approx 1$ or $\Upsilon \approx 1$ and for flows with a small core, when $\chi << 1$ or $\Upsilon << 1$, one can use the following approximations:

– for a large core

$$F_{(n)}^{(m)} \approx \chi^{\frac{m+2n}{n}} \sum_{i=0}^{m} (-1)^i \frac{n}{m+2n-i} C_m^i = \chi^{\frac{m+2n}{n}} \frac{(-1)^m}{2C_{m+2n}^m},$$

$$F_{(n)}^{(n)} \approx \chi^3 \sum_{i=0}^{n} (-1)^i \frac{n}{3n-i} C_n^i = \chi^3 \frac{(-1)^n}{2C_{3n-1}^n}, \tag{8.21}$$

$$T_{(n)}^{(m)} \approx \Upsilon^{\frac{m+3n}{n}} \sum_{i=0}^{m} (-1)^i \frac{n}{m+3n-i} C_m^i = \Upsilon^{\frac{m+3n}{n}} \frac{(-1)^m}{3C_{m+3n}^m},$$

Cont. (8.21)

$$T_{(n)}^{(n)} \approx \Upsilon^4 \sum_{i=0}^{n} (-1)^i \frac{n}{4n-i} C_n^i = \Upsilon^4 \frac{(-1)^n}{4C_{4n-1}^n};$$

- for a small core

$$F_{(n)}^{(m)} \approx \frac{n}{m+2n}\left[1-\frac{m(m+2n)}{m+2n-1}\chi^{\frac{1}{n}}\right],$$

(8.22)

$$T_{(n)}^{(m)} \approx \frac{n}{m+3n}\left[1-\frac{m(m+3n)}{m+3n-1}\Upsilon^{\frac{1}{n}}\right].$$

Thus, we may write for the pressure distribution in capillaries of variable cross-sections:

– in the capillary fissure

$$\Delta p = 2\left[\frac{\mu Q_f}{2F_{(n)}^{(m)}}\right]^{n/m} \int_0^{h/2} \frac{dy}{f_c^{\frac{m+2n}{m}}}, \tag{8.23}$$

– in the capillary tube

$$\Delta p = 4\left[\frac{\mu Q_t}{\pi T_{(n)}^{(m)}}\right]^{n/m} \int_0^{h/2} \frac{dy}{r_c^{\frac{m+3n}{m}}}. \tag{8.24}$$

Introducing here the expressions (2.1) and (2.2) and taking into account the considerations of Section 5, we can write that:

– for the capillary fissure

$$\Delta p_f = 2\left[\frac{\mu Q_f}{2F_{(n)}^{(m)}}\right]^{n/m} \frac{h\left(2^{\alpha_f-1}-1\right)}{f_o^{\frac{m+2n}{m}}\left(\alpha_f-1\right)}, \tag{8.25}$$

$$Q_f = F_{(n)}^{(m)} \frac{2f_o^{\frac{m+2n}{n}}}{(m+2)\mu}\left[\frac{\alpha_f-1}{2\left(2^{\alpha_f-1}-1\right)}\right]^{\frac{m}{n}}\left(\frac{\Delta p_f}{h}\right)^{\frac{m}{n}}, \tag{8.26}$$

$$\upsilon_f = F_{(n)}^{(m)} \frac{f_o^{\frac{m+n}{n}}}{\mu} \left[\frac{\alpha_f - 1}{2\left(2^{\alpha_f - 1} - 1\right)} \right]^{\frac{m}{n}} \left(-\frac{dp}{dy} \right)^{\frac{m}{n}} ; \tag{8.27}$$

– for the capillary tube

$$\Delta p_t = 4 \left[\frac{\mu Q_t}{\pi T_{(n)}^{(m)}} \right]^{n/m} \frac{h\left(2^{\alpha_t - 1} - 1\right)}{r_o^{\frac{m+3n}{m}} \left(\alpha_t - 1\right)} , \tag{8.28}$$

$$Q_t = T_{(n)}^{(m)} \frac{\pi r_o^{\frac{m+3n}{n}}}{2^{\frac{m}{n}} \mu} \left[\frac{\alpha_t - 1}{2\left(2^{\alpha_t - 1} - 1\right)} \right]^{\frac{m}{n}} \left(\frac{\Delta p_t}{h} \right)^{\frac{m}{n}} , \tag{8.29}$$

$$\upsilon_t = T_{(n)}^{(m)} \frac{r_o^{\frac{m+n}{n}}}{2^{\frac{m}{n}} \mu} \left[\frac{\alpha_t - 1}{2\left(2^{\alpha_t - 1} - 1\right)} \right]^{\frac{m}{n}} \left(-\frac{dp}{dy} \right)^{\frac{m}{n}} \tag{8.30}$$

where

$$\alpha_f = \frac{n_c\left(m + 2n\right)}{m}, \qquad \alpha_t = \frac{n_c\left(m + 3n\right)}{m} . \tag{8.31}$$

Note that these formulae for $n = 1$ reduce to the formulae adequate to the power-law fluids.

9. Some remarks on pore network models (PNMs)

In these models, the soil pore space is modelled by a discrete network of pores that are connected by throats, which may be prismatic or non-prismatic. Pore bodies are considered to be either identical cylinders or identical spheres which are spaced equally space h in all body domain (Fig.3). By writing a mass-balance equation (being a counterpart of the first Kirchhoff law) for a junction, an algebraic equation relating a considered node to its surrounding nodes is obtained (Fig.4).

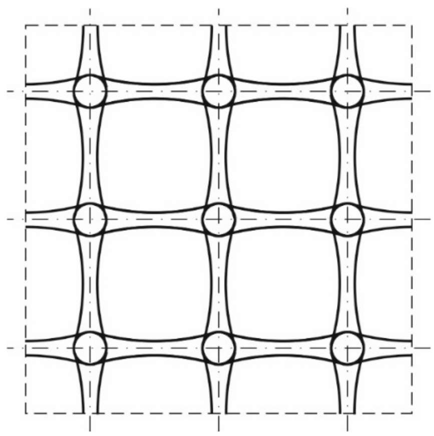

Fig.3. A two-dimensional (2D) pore network.

Applying this scheme to all nodes, a system of linear algebraic equations is obtained. To solve this system one of the well known methods from the theory of finite differences can be used. To illustrate the procedure let us consider at first the flow of a Newtonian fluid.
The mass balance equation for an arbitrary interior node "i" has a form (see Fig.4)

$$\sum_{j=1}^{4} Q_{ij} = 0 \tag{9.1}$$

where

$$Q_{ij} = C'_{ij}\left(\frac{\Delta p}{h}\right) = C_{ij}\left(\Delta p\right) = C_{ij}\left(p_i - p_j\right). \tag{9.2}$$

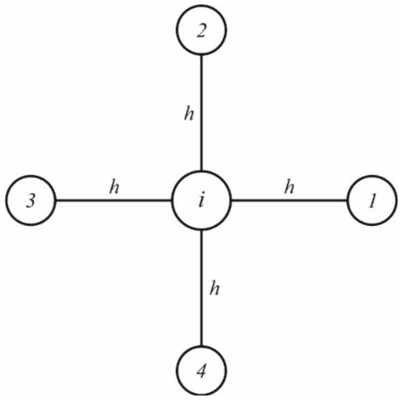

Fig.4. An arbitrary interior node "i".

We have (see formulae (3.8) or (3.10)), respectively

$$C_{ij} = \frac{f_o^3}{3\mu h}\left[\frac{3n_c - 1}{2^{3n_c - 1} - 1}\right] \qquad \text{for capillary fissures,}$$

$$C_{ij} = \frac{\pi r_o^4}{16\mu h}\left[\frac{4n_c - 1}{2^{4n_c - 1} - 1}\right] \qquad \text{for capillary tubes.} \tag{9.3}$$

In the case of a homogenous and isotropic porous medium, $C_{ij}s$ would all be the same, and (Eq.9.1) reduces to

$$4p_i - p_1 - p_2 - p_3 - p_4 = 0. \tag{9.4}$$

This equation presents the discretized form of a steady flow equation through a saturated porous medium and it is the same as the discretized form obtained by central finite differences. The same procedure may be followed (for capillary tubes only) for a typical node in a $3D$ domain.

Let us consider now the flow of a power-law fluid. The mass balance equation leads to the expression

$$\sum_{j=1}^{4} Q_{ij}^{(m)}\left(p_i - p_j\right)^m = 0 \tag{9.5}$$

where (see formulae (5.8) and (5.11))

$$C_{ij}^{(m)} = \frac{2 f_o^{m+2}}{(m+2) h^m \mu} \left[\frac{\alpha_f - 1}{2\left(2^{\alpha_f - 1} - 1\right)} \right]^m \qquad \text{for capillary fissures,}$$

$$C_{ij}^{(m)} = \frac{\pi r_o^{m+3}}{2^m (m+3) h^m \mu} \left[\frac{\alpha_t - 1}{2\left(2^{\alpha_t - 1} - 1\right)} \right]^m \qquad \text{for capillary tubes.} \tag{9.6}$$

If $p_j << p_i$, then

$$\left(p_i - p_j\right)^m \approx p_i^m \left(1 - m\frac{p_j}{p_i}\right) \tag{9.7}$$

and applying Eq.(9.5) we find that

$$4 p_i - m\left(p_1 + p_2 + p_3 + p_4\right) = 0\,. \tag{9.8}$$

If $p_j \approx p_i$, then

$$\left(p_i - p_j\right)^m = p_i^m \left(1 - \frac{p_j}{p_i}\right)^m = p_i^m \left[1 - \sum_{k=1}^{m} (-1)^{k+1} C_k^m \left(\frac{p_j}{p_i}\right)^k\right]. \tag{9.9}$$

Applying Eq.(9.5) we have

$$4P_i - P_1 - P_2 - P_3 - P_4 = 0 \tag{9.10}$$

where

$$P_i = p_i^m, \qquad P_j = \sum_{k=1}^{m} (-1)^{k+1} C_k^m p_j^k p_i^{m-k}\,. \tag{9.11}$$

Similar equations may be obtained for the flow of Shulman fluids.

Applying the procedures presented above it may obtain the computational stencils for more complex models of fluids can be obtained. Let us consider the flow of DeHaven fluids. Writing the mass balance equation, we have (see formulae (6.14) and (6.19))

$$\sum_{j=1}^{4} \left[C_{ij}\left(p_i - p_j\right) + M_{ij}\left(p_i - p_j\right)^{n_i+1} \right] = 0 \tag{9.12}$$

or

$$\sum_{j=1}^{4} \left[\left(p_i - p_j\right) + \frac{M_{ij}}{C_{ij}}\left(p_i - p_j\right)^{n_i+1} \right] = 0 \tag{9.13}$$

where

$$M_{ij} = \frac{k_i f_o^{n_i+3}}{2^{n_i}\left(n_i+3\right)\mu}\left[\frac{\alpha_f - 1}{2^{\alpha_f - 1} - 1}\right]^{n_i+1} \qquad \text{for capillary fissures,}$$

(9.14)

$$M_{ij} = \frac{\pi k_i r_o^{n_i+4}}{4^{n_i+1}\left(n_i+4\right)\mu}\left[\frac{\alpha_t - 1}{2^{\alpha_t - 1} - 1}\right]^{n_i+1} \qquad \text{for capillary tubes.}$$

Assuming, e.g., that $p_j << p_i$, we may write

$$\frac{4\left(1+\beta_i\right)}{1+\left(n_i+1\right)\beta_i} p_i - \left(p_1 + p_2 + p_3 + p_4\right) = 0 \tag{9.15}$$

where

$$\beta_i = \frac{M_{ij} p_i^{n_i}}{C_{ij}}. \tag{9.16}$$

One may obtain similarly the mass balance equation for the case when $p_j \approx p_i$.

Note that in each case of fluid models under consideration the final mass balance equation is linear or quasi-linear. To solve these equations it is convenient-to use an iterative procedure well known in the theory of finite differences.

Conclusions

In this paper, an approximate mathematical method for obtaining the analytical relations between pressure drops and volumetric flow rates in symmetrically corrugated fissures and tubes is presented and applied to the flows of Newtonian, polar, power-law, DeHaven, Sisko and Shulmanian fluids.

Taking into account the considerations on the flows through rectilinear capillaries of constant cross-sections , a general method to describe the flows through convergent-divergent (in general) capillaries with exponential variability of cross-sections was presented.

Formulae for pressure drops, volumetric flow rates and flow velocities through thin porous layers for all cited above fluids were obtained. Finally, some remarks on the pore network models (PNMs) for Newtonian and selected non-Newtonian fluids were made. These models may be used to transform the differential equations governing the flow through porous media into a set of algebraic equations similar to these ones known from finite differences.

Nomenclature

D_o – outside half thickness or outside radius of a capillary

$F_{(n)}^{(m)}$ – functions of a relative core flow thickness in Shulmanian fluids

f_c – half thickness of a constant cross-section capillary fissure or current half thickness of a variable cross-section capillary fissure

f_o – outside half thickness of a variable cross-section capillary fissure

f_0 – half thickness of a core flow through the capillary fissure in Shulmanian fluids
h – thickness of a porous layer
k – vortex (coupling) viscosity of a micropolar fluid
k_i – material coefficients for DeHaven fluids
m, n – flow behaviour indices for Shulmanian fluids
n_i – flow behaviour indices for DeHaven and Sisko fluids
n_c – exponent of a capillary convergence
p – pressure
Δp_f – pressure drop in a capillary fissure
Δp_t – pressure drop in a capillary tube
Q – volumetric flow rate
Q_f – volumetric flow rate through the unity width of a capillary fissure
Q_t – volumetric flow rate through a capillary tube
r_c – radius of a constant cross-section capillary tube or current radius of a variable cross-section capillary tube
r_o – outside radius of a variable cross-section capillary tube
r_0 – radius of a core flow through the capillary tube in Shulmanian fluids
β – permeability of a porous medium
η – material coefficient associated with couple-stresses
μ – shear or plastic viscosity
μ_i – material coefficients for Sisko fluids
τ_0 – yield shear stress
τ_w – wall shear stress
χ, Υ – relative thickness/radius of a core flow through the capillary fissure or tube in Shulmanian fluids

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