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BASIC FLOWS OF GENERALIZED SECOND GRADE FLUIDS BASED ON A SISKO MODEL

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The present investigation is concerned with basic flows of generalized second grade fluids based on a Sisko fluid. After formulation of the general equations of motion three simple flows of viscoplastic fluids of a Sisko type or fluids similar to them are considered. These flows are: Poiseuille flow in a plane channel, Poiseuille flow in a circular pipe and rotating Couette flow between two coaxial cylinders. After presentation the Sisko model one was presented some models of fluids similar to this model. Next it was given the solutions of equations of motion for three flows mentioned above.

Key words: Sisko fluids, similar fluids, simple flows.

1. Introduction

In this paper we will consider the group of pseudoplastic fluids whose viscosity displays a non-linear relationship between the shear stress and the shear strain rate. Here the constitutive equations consider the shear stress as a non-linear function of the shear strain rate. One of more general models of this kind of fluids is a Sisko model [1, 2]. There are a few simple solutions of the equations of motion for the flows of a Sisko fluid. The first were provided by Na and Hansen [3] and Bahrami *et al.* [4]; thereafter by Wang *et al.* [5], Hayat *et al.* [6], Khan *et al.* [7], Mekheimer and Kot [8], Khan and Shahzad [9], Akbar [10], Walicka [11].

The flows of fluids, whose models are similar to the Sisko model, were also studied by numerous rheologists; the researchers analysing the peristaltic flows should be mentioned here, for example: Akbar *et al.* [12] who analysed the flow of a Prandtl fluid, Nadeen [13] who analysed the flow of a tangent hyperbolic fluid model, Ellahi *et al.* [14] who analysed the flow of a Carreau fluid, etc.

In what follows we will present simple flows of generalized second grade fluids based on the Sisko model and similar models (see Table 1).
To this consider the other similar models of pseudoplastic fluid given in the second column of Tab.1. Note that for suitably selected material coefficients these models can be presented in a simple unified form

$$\tau = \left(\mu_0 + \mu_i \left|\dot{\gamma}\right|^{n_i}\right). \tag{1.1}$$

To find three-dimensional forms of the stress tensor $\boldsymbol{T}$, corresponding to the above given one-dimensional form of τ (called a constitutive relation), we may use the generalization of the Prager-Oldroyd method (Prager, [15]) applied to a Bingham fluid. This generalization can be found in (Walicka, [16, 17]) and its basis is as follows:
if

$$\tau = f\left(\dot{\gamma}\right)\dot{\gamma}, \tag{1.2}$$

then

$$(\boldsymbol{T})_{ij} = T_{ij} = -p\delta_{ij} + 2f(A)D_{ij} = -p\delta_{ij} + f(A)(\boldsymbol{A}_1)_{ij}, \tag{1.3}$$

where p is the pressure and

$$A = \left[\frac{1}{2}tr\left(\boldsymbol{A}_1^2\right)\right]^{\frac{1}{2}}; \tag{1.4}$$

here A is a square root from the second invariant of $\boldsymbol{A}_1$ [18].

Table.1. Model of fluids similar to the Sisko fluid model [18].

Author(s)	Model	Reduced model	μ_i	Comments
$n_i = n$				"$n+1$" power models
Sisko	$\tau = \left[\mu_0 + \mu\lvert\dot{\gamma}\rvert^n\right]\dot{\gamma}$	–	μ	
Carreau-Yasuda	$\tau = \left\{\mu_\infty + \dfrac{\mu_0 - \mu_\infty}{\left[1+(\kappa\dot{\gamma})^n\right]^{\frac{\alpha}{n}}}\right\}\dot{\gamma}$	$\tau \approx \left[\mu_0 - \dfrac{\alpha\mu}{n}(\kappa\dot{\gamma})^n\right]\dot{\gamma}$; $\mu = \mu_0 - \mu_\infty$	$-\dfrac{\alpha\mu\kappa^n}{n}$	Cross model for $\alpha = n$ Williamson-Moore model for $\alpha = n = 1$
Elsharkawy-Hamrock	$\tau = \dfrac{\mu_0\dot{\gamma}}{\left[1+(\kappa\dot{\gamma})^n\right]^{\frac{1}{n}}}$	$\tau \approx \left[\mu_0 - \dfrac{\mu_0}{n}(\kappa\dot{\gamma})^n\right]\dot{\gamma}$	$-\dfrac{\mu_0\kappa^n}{n}$	
$n_i = 2$				"Cubic" models
Prandtl	$\tau = \mu_0\dfrac{\arcsin(\kappa\dot{\gamma})}{(\kappa\dot{\gamma})}\dot{\gamma}$	$\tau \approx \left[\mu_0 + \dfrac{\mu_0}{6}(\kappa\dot{\gamma})^2\right]\dot{\gamma}$	$\dfrac{\mu_0\kappa^2}{6}$	
Eyring-Sutterby	$\tau = \mu_0\left[\dfrac{\operatorname{ar\,sinh}(\kappa\dot{\gamma})}{\kappa\dot{\gamma}}\right]^n\dot{\gamma}$	$\tau \approx \left[\mu_0 - \dfrac{n\mu_0}{6}(\kappa\dot{\gamma})^2\right]\dot{\gamma}$	$-\dfrac{n\mu_0\kappa^2}{6}$	Prandtl-Eyring model for $n = 1$
Sutterby	$\tau = \left\{\mu_\infty + (\mu_0 - \mu_\infty)\times \left[\dfrac{\operatorname{ar\,sinh}(\kappa\dot{\gamma})}{\kappa\dot{\gamma}}\right]^n\right\}\dot{\gamma}$	$\tau \approx \left[\mu_0 + \dfrac{n\mu}{6}(\kappa\dot{\gamma})^2\right]\dot{\gamma}$; $\mu = \mu_0 - \mu_\infty$	$-\dfrac{n\mu\kappa^2}{6}$	Powell-Eyring model for $n = 1$
Gecim-Winer	$\tau = \mu_0\left[\dfrac{\tanh(\kappa\dot{\gamma})}{(\kappa\dot{\gamma})}\right]\dot{\gamma}$	$\tau \approx \left[\mu_0 - \dfrac{\mu_0}{3}(\kappa\dot{\gamma})^2\right]\dot{\gamma}$	$-\dfrac{\mu_0\kappa^2}{3}$	
$n_i = 1$				"Quadratic" models
Bair-Winer	$\tau = \mu_0\left(\dfrac{1-e^{-\kappa\dot{\gamma}}}{\kappa\dot{\gamma}}\right)\dot{\gamma}$	$\tau \approx \left[\mu_0 - \dfrac{\mu_0}{2}(\kappa\dot{\gamma})\right]\dot{\gamma}$	$-\dfrac{\mu_0\kappa}{3}$	

2. Equations of motion of the unified Sisko fluid model

The general equations of motion of a viscous fluid in a three-dimensional form are as follows:

– equation of continuity

$$\mathrm{div}\,\mathbf{v} = 0\,, \tag{2.1}$$

– equation of momentum

$$\rho\frac{d\mathbf{v}}{dt} = \mathrm{div}\boldsymbol{T}\,, \qquad \boldsymbol{T} = -p\boldsymbol{1} + \boldsymbol{\Lambda} \tag{2.2}$$

or

$$\rho\frac{d\mathbf{v}}{dt} = -\nabla p + \mathrm{div}\,\boldsymbol{\Lambda}\,, \tag{2.3}$$

here

$$\boldsymbol{\Lambda} = \left[\mu_0 + \mu_i\left(A\right)^{n_i}\right]\ \boldsymbol{A}_1\,. \tag{2.4}$$

Many fluids of engineering interest appear to exhibit viscoelastic behaviour. Most popular are second grade fluids [16÷20].

The constitutive relation for the second grade fluids is given as follows (Rivlin and Ericksen, [19])

$$\boldsymbol{T} = -p\boldsymbol{I} + \mu\boldsymbol{A}_1 + \alpha\boldsymbol{A}_1^2 + \beta\boldsymbol{A}_2\,, \tag{2.5}$$

where p is the pressure, μ is the coefficient of viscosity, α,β are material moduli, $\boldsymbol{A}_1$ and $\boldsymbol{A}_2$ are the first two Rivlin-Ericksen tensors defined by

$$\boldsymbol{A}_1 = \boldsymbol{2D} = \boldsymbol{L} + \boldsymbol{L}^T\,, \quad \boldsymbol{A}_1^2 = \boldsymbol{A}_1\cdot\boldsymbol{A}_1 = \left(\boldsymbol{L} + \boldsymbol{L}^T\right)^2, \quad \boldsymbol{L} = \mathrm{grad}\,\mathbf{v}\,,$$
$$\boldsymbol{A}_2 = \dot{\boldsymbol{A}}_1 + \boldsymbol{A}_1\boldsymbol{L} + \boldsymbol{L}^T\boldsymbol{A}_1 \quad \text{or} \quad \boldsymbol{A}_2 = \mathrm{grad}\,\boldsymbol{a} + \left(\mathrm{grad}\,\boldsymbol{a}\right)^T + 2\boldsymbol{L}^T\boldsymbol{L} \tag{2.6}$$

and

$$\boldsymbol{a} = \frac{d\mathbf{v}}{dt} = \frac{\partial\mathbf{v}}{\partial t} + \mathbf{v}\boldsymbol{L}\,, \tag{2.7}$$

where $\mathbf{v}$ is the velocity vector, $\boldsymbol{a}$ is the acceleration vector and $\dot{(\cdot)}$ represents the material derivative with respect to time.

To obtain a model that are does exhibit both pseudoplastic and viscoelastic behaviour we propose the following two constitutive equations for generalized second grade fluids (Walicki and Walicka, [20], Walicka [16÷18]):

– for model I

$$\boldsymbol{T} = -p\boldsymbol{1} + M_i\boldsymbol{A}_1 + \alpha_1\boldsymbol{A}_1^2 + \beta_1\boldsymbol{A}_2\,, \tag{2.8}$$

– for model II

$$\boldsymbol{T} = -p\boldsymbol{1} + M_i\left(\boldsymbol{A}_1 + \alpha_2\boldsymbol{A}_1^2 + \beta_2\boldsymbol{A}_2\right), \tag{2.9}$$

The viscosity function M_i is given as follows:

$$M_i = \mu_0 + \mu_i\left(A\right)^{n_i}. \tag{2.10}$$

3. Poiseuille flow in a plane channel

Let us a consider the steady laminar fully developed flow of a generalized second grade Sisko fluid between two horizontal parallel plates (Fig.1). The flow takes place along to a pressure along the plates located at $y = -h$ and $y = +h$, respectively.

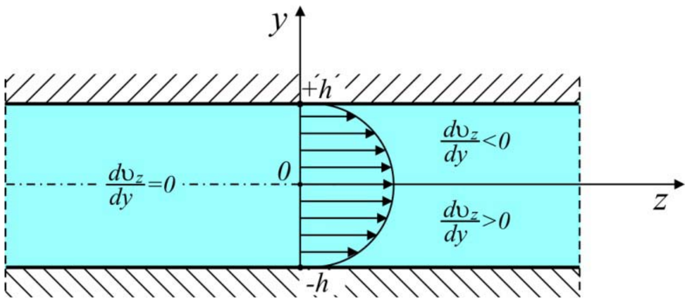

Fig.1. Channel between two parallel plates.

The flow field has the form

$$\mathbf{v} = \upsilon_z\left(y\right)\boldsymbol{k}, \qquad p = p\left(y,z\right). \tag{3.1}$$

Substituting Eqs (3.1) into Eq.(2.2), we obtain

$$\frac{\partial}{\partial z}\left[-p + \alpha_j M_i^{j-1}\left(\frac{d\upsilon_z}{dy}\right)^2\right] + \frac{\partial}{\partial y}\left[M_i\,\frac{d\upsilon_z}{dy}\right] = 0\,, \tag{3.2}$$

$$\frac{\partial}{\partial y}\left[-p + \left(\alpha_j + 2\beta_j\right)M_i^{j-1}\left(\frac{d\upsilon_z}{dy}\right)^2\right] = 0, \tag{3.3}$$

where $j = 1$ for model *I* and $j = 2$ for model *II*.

Let us define a modified pressure $\hat{p}^{(j)}$ through

$$\hat{p}^{(j)} = p - \left(\alpha_j + 2\beta_j\right)M_i^{j-1}\left(\frac{d\upsilon_z}{dy}\right)^2, \tag{3.4}$$

then from Eqs.(2.3) and (3.2), we obtain

$$\frac{d\hat{p}^{(j)}}{dz}=\frac{d\Lambda_{zy}}{dy}, \tag{3.5}$$

where

$$\Lambda_{zy}=\left(\mu_0+\mu_i\left|\frac{d\upsilon_z}{dy}\right|^{n_i}\right)\frac{d\upsilon_z}{dy}. \tag{3.6}$$

Upon introducing Eq.(3.6) into Eq.(3.5), we have

$$\frac{d\hat{p}^{(j)}}{dz}=\frac{d}{dy}\left[\left(\mu_0+\mu_i\left|\frac{d\upsilon_z}{dy}\right|^{n_i}\right)\frac{d\upsilon_z}{dy}\right]$$

or

$$\frac{1}{\mu_0}\frac{d\hat{p}^{(j)}}{dz}=\frac{d^2\upsilon_z}{dy^2}+\frac{\mu_i\left(n_i+1\right)}{\mu_0}\left|\frac{d\upsilon_z}{dy}\right|^{n_i}\frac{d^2\upsilon_z}{dy^2}. \tag{3.7}$$

The boundary conditions on the plates are stated as follows

$$\upsilon_z=0 \qquad \text{for} \qquad y=\pm h. \tag{3.8}$$

Let us develop υ_z into a power series

$$\upsilon_z=\beta^0\upsilon_0+\beta^1\upsilon_1+\beta^2\upsilon_2+\ldots, \tag{3.9}$$

where

$$\beta=\frac{\mu_i\left(n_i+1\right)}{\mu_0}<1. \tag{3.10}$$

Putting Eq.(3.9) into Eq.(3.7) and retaining only the two first terms, we find

$$\frac{1}{\mu_0}\frac{d\hat{p}^{(j)}}{dz}=\frac{d^2\upsilon_0}{dy^2}+\beta\left[\frac{d^2\upsilon_1}{dy^2}+\left(\left|\frac{d\upsilon_0}{dy}\right|^{n_i}\right)\frac{d^2\upsilon_0}{dy^2}\right]. \tag{3.11}$$

After equating the like powers of β we will obtain two equations:

– for β^0:

$$\frac{1}{\mu_0}\frac{d\hat{p}^{(j)}}{dz}=\frac{d^2\upsilon_0}{dy^2}, \tag{3.12}$$

– for β^1:

$$\frac{d^2\upsilon_1}{dy^2} = -\left(\left|\frac{d\upsilon_0}{dy}\right|^{n_i}\right)\frac{d^2\upsilon_0}{dy^2}. \tag{3.13}$$

Solving these equations we will obtain

$$\upsilon_0 = \frac{h^2 - y^2}{2}\left(-\frac{1}{\mu_0}\frac{d\hat{p}^{(j)}}{dz}\right), \tag{3.14}$$

and

$$\upsilon_1 = \frac{h^{n_i+2} - |y|^{n_i+2}}{(n_i+1)(n_i+2)}\left(-\frac{1}{\mu_0}\frac{d\hat{p}^{(j)}}{dz}\right)^{n_i+1}. \tag{3.15}$$

Finally:

$$\upsilon_z = \frac{h^2 - y^2}{2}\left(-\frac{1}{\mu_0}\frac{d\hat{p}^{(j)}}{dz}\right) - \frac{\mu_i}{\mu_0}\frac{h^{n_i+2} - |y|^{n_i+2}}{(n_i+2)}\left(-\frac{1}{\mu_0}\frac{d\hat{p}^{(j)}}{dz}\right)^{n_i+1}. \tag{3.16}$$

The flow rate Q is defined as

$$Q = 2\int_0^h \upsilon_z dy = \frac{3h^3}{3}\left(-\frac{1}{\mu_0}\frac{d\hat{p}^{(j)}}{dz}\right)\left[1 - \frac{3\mu_i h^{n_i}}{\mu_0(n_i+3)}\left(-\frac{1}{\mu_0}\frac{d\hat{p}^{(j)}}{dz}\right)^{n_i}\right]. \tag{3.17}$$

Introducing the notation

$$X = -\frac{1}{\mu_0}\frac{d\hat{p}^{(j)}}{dz} \tag{3.18}$$

one can rewrite Eq.(3.17) in the form

$$\frac{3\mu_i h^{n_i}}{\mu_0(n_i+3)}X^{n_i+1} - X + \frac{3Q}{2h^3} = 0. \tag{3.19}$$

Denoting its solution by X_s, we have

$$\frac{d\hat{p}^{(j)}}{dz} = -\mu_0 X_s \tag{3.20}$$

hence

$$\hat{p}^{(j)} = C - \mu_0 X_s z \tag{3.21}$$

and

$$\frac{d\upsilon_z}{dy} = -yX_s + \frac{\mu_i}{\mu_0}|y|^{n_i+1} X_s^{n_i+1}. \tag{3.22}$$

The final formula for the pressure distribution is as follows

$$p = \hat{p}^{(j)} + \left(\alpha_j + 2\beta_j\right) M_i^{j-1} \left(\frac{d\upsilon_z}{dy}\right)^2. \tag{3.23}$$

Note that for the regular Sisko fluid

$$p = \hat{p}^{(j)}. \tag{3.24}$$

4. Poiseuille flow through a circular pipe

Let us consider the steady laminar flow of a generalized second grade Sisko fluid in a circular pipe of radius R (Fig.2). We are concerned about the velocity field in the form of:

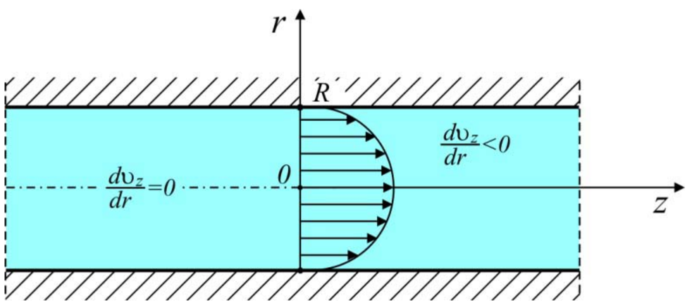

Fig.2. Geometry of a circular pipe.

$$\upsilon_z = 0, \qquad \upsilon_\vartheta = 0, \qquad \upsilon_z = \upsilon_z(r). \tag{4.1}$$

The equations of motion are now

$$\frac{\partial p}{\partial r} = \left(\alpha_j + 2\beta_j\right) \frac{1}{r} \frac{d}{dr} \left[r M_i^{j-1} \left(\frac{d\upsilon_z}{dr}\right)^2 \right], \tag{4.2}$$

$$\frac{\partial p}{\partial z} = \frac{1}{r} \frac{d}{dr} \left[r M_i \frac{d\upsilon_z}{dr} \right], \tag{4.3}$$

where $j = 1$ for model *I* and $j = 2$ for model *II*.

Here we have

$$M_i = \left[\mu_0 + \mu_i \left(-\frac{d\upsilon_z}{dr} \right)^{n_i} \right], \tag{4.4}$$

therefore Eq.(4.3) can be presented as

$$\frac{\partial p}{\partial z} = \frac{1}{r} \frac{d\left(r\Lambda_{rz} \right)}{dr}, \tag{4.5}$$

where

$$\Lambda_{rz} = M_i \frac{d\upsilon_z}{dr}; \tag{4.6}$$

hence

$$\frac{r}{\mu_0} \frac{\partial p}{\partial z} = \frac{d}{dr}\left(r \frac{d\upsilon_z}{dr} \right) - \frac{\mu_i}{\mu_0} \frac{d}{dr}\left[r \left(-\frac{d\upsilon_z}{dr} \right)^{n_i+1} \right]. \tag{4.7}$$

The boundary conditions are

$$\upsilon_z = 0, \quad \text{for} \quad r = R, \qquad \frac{d\upsilon_z}{dr} = 0, \quad \text{for} \quad r = 0. \tag{4.8}$$

Develop υ_z into a power series:

$$\upsilon_z = \upsilon_0 + \beta\upsilon_1 + \beta^2\upsilon_2 + \ldots, \tag{4.9}$$

where

$$\beta = \frac{\mu_i}{\mu_0} \tag{4.10}$$

and on introducing this series into Eq.(4.7), we will obtain for the two first powers of β the following equations:

– for β^0

$$\frac{d}{dr}\left(r \frac{d\upsilon_0}{dr} \right) = \frac{r}{\mu_0} \frac{\partial p}{\partial z}, \tag{4.11}$$

– for β^1

$$\frac{d}{dr}\left(r \frac{d\upsilon_1}{dr} \right) = \frac{d}{dr}\left[r \left(-\frac{d\upsilon_0}{dy} \right)^{n_i+1} \right]. \tag{4.12}$$

Solving these equations we have, respectively

$$\upsilon_0 = \frac{R^2 - r^2}{4\mu_0}\left(-\frac{\partial p}{\partial z}\right), \tag{4.13}$$

and

$$\upsilon_1 = -\frac{R^{n_i+2} - r^{n_i+2}}{\left(2\mu_0\right)^{n_i+1}\left(n_i+2\right)}\left(-\frac{\partial p}{\partial z}\right)^{n_i+1}. \tag{4.14}$$

Finally, according to Eq.(4.9), there is

$$\upsilon_z = \frac{R^2 - r^2}{4\mu_0}\left(-\frac{\partial p}{\partial z}\right) - \frac{\mu_i}{\mu_0}\frac{R^{n_i+2} - r^{n_i+2}}{\left(2\mu_0\right)^{n_i+1}\left(n_i+2\right)}\left(-\frac{\partial p}{\partial z}\right)^{n_i+1}. \tag{4.15}$$

The flow rate is equal to

$$Q = 2\pi\int_0^R \upsilon_z\left(r\right) r dr = \frac{\pi R^4}{4}\left(-\frac{1}{2\mu_0}\frac{\partial p}{\partial z}\right)\left[1 - \frac{4\mu_i R^{n_i}}{\left(n_i+4\right)}\left(-\frac{1}{2\mu_0}\frac{\partial p}{\partial z}\right)^{n_i}\right]. \tag{4.16}$$

Introducing the notation

$$Y = -\frac{1}{2\mu_0}\frac{\partial p}{\partial z} \tag{4.17}$$

one can rewrite Eq.(4.16) in the form

$$\frac{4\mu_i R^{n_i}}{\mu_0\left(n_i+4\right)} Y^{n_i+1} - Y + \frac{4Q}{\pi R^4} = 0. \tag{4.18}$$

Denoting its solution by Y_s, we have

$$\frac{\partial p}{\partial z} = -2\mu_0 Y_s \tag{4.19}$$

and

$$\frac{d\upsilon_z}{dr} = rY_s\left[1 + \left(\frac{\mu_i}{\mu_0}\right) r^{n_i} Y_s^{n_i}\right]. \tag{4.20}$$

The pressure distribution is now given as follows

$$p(r,z) = C - 2\mu_0 Y_s z + (\alpha_j + 2\beta_j)\int \frac{1}{r}\frac{d}{dr}\left[r M_i^{j-1}\left(\frac{d\upsilon_z}{dr}\right)^2\right] dr \tag{4.21}$$

or

$$p(r,z) = C - 2\mu_0 Y_s z + (\alpha_j + 2\beta_j) J_j(r,z), \tag{4.22}$$

where

$$J_1(r,z) = \frac{3}{2}(rY_s)^2 + \frac{2(n_i+3)}{(n_i+2)}\left(\frac{\mu_i}{\mu_0}\right)(rY_s)^{n_i+2} + \frac{2n_i+3}{2n_i+2}\left(\frac{\mu_i}{\mu_0}\right)^2 (rY_s)^{2n_i+2} \tag{4.23}$$

or

$$J_2(r,z) = \mu_0 J_1(r,z) + (-1)^{n_i+2}\mu_i \sum_{i=0}^{n_i+2} \frac{(i+1)n_i+3}{(i+1)n_i+2} C_{n_i+2}^{i} \left(\frac{\mu_i}{\mu_0}\right)^i (rY_s)^{(i+1)n_i+2}. \tag{4.24}$$

Note that for a regular Sisko fluid there is

$$p = p(z). \tag{4.25}$$

5. Rotating Couette flow between two coaxial cylinders

The fluid flow configuration is shown in Fig.3; the inner cylinder of radius R_i rotates with a constant angular velocity ω and the outer cylinder of radius R_o is fixed. The flow field of the fluid is given by

$$\upsilon_r = 0, \qquad \upsilon_\vartheta = \upsilon_\vartheta(r), \qquad \upsilon_z = 0. \tag{5.1}$$

The equations of motion take the form

$$\frac{dp}{dr} = \rho\frac{\upsilon_\vartheta^2}{r} + (\alpha_j + 2\beta_j)\frac{d}{dr}M_i^{j-1}\left[r\frac{d}{dr}\left(\frac{\upsilon_\vartheta}{r}\right)\right]^2 + 2\beta_j\frac{M_i^{j-1}}{r}\left[r\frac{d}{dr}\left(\frac{\upsilon_\vartheta}{r}\right)\right]^2, \tag{5.2}$$

here $j = 1$ for model *I* and $j = 2$ for model *II*;

$$\frac{d}{dr}\left(r^2\Lambda_{r\vartheta}\right) = 0, \tag{5.3}$$

where

$$\Lambda_{r\vartheta} = M_i\left[r\frac{d}{dr}\left(\frac{\upsilon_\vartheta}{r}\right)\right] \tag{5.4}$$

and

$$M_i = \mu_0 + \mu_i \left[-r \frac{d}{dr}\left(\frac{\upsilon_\vartheta}{r} \right) \right]^{n_i}; \tag{5.5}$$

finally,

$$\Lambda_{r\vartheta} = \left\{ \mu_0 + \mu_i \left[-r \frac{d}{dr}\left(\frac{\upsilon_\vartheta}{r} \right) \right]^{n_i} \right\} \left[r \frac{d}{dr}\left(\frac{\upsilon_\vartheta}{r} \right) \right]. \tag{5.6}$$

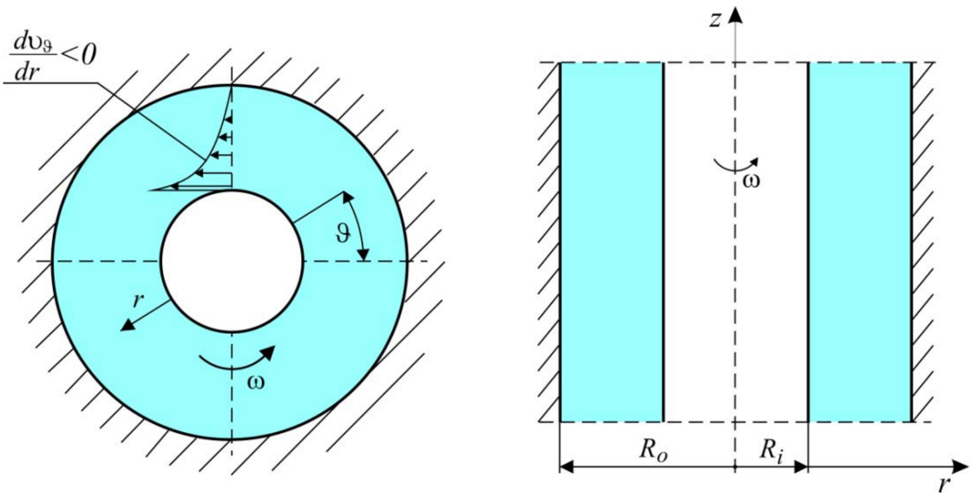

Fig.3. Geometry of the rotational flow between cylindrical surfaces.

The boundary conditions for velocity are

$$\upsilon_\vartheta = R_i \omega, \quad \text{for} \quad r = R_i,$$
$$\upsilon_\vartheta = 0, \quad \text{for} \quad r = R_o. \tag{5.7}$$

Upon integration of Eq.(5.3), we will obtain

$$\Lambda_{r\vartheta} = \frac{C_1}{r^2}. \tag{5.8}$$

This result introduced into Eq.(5.6), will yield

$$\frac{C_1}{\mu_0 r^2} = r \frac{d}{dr}\left(\frac{\upsilon_\vartheta}{r} \right) - \frac{\mu_i}{\mu_0} \left[-r \frac{d}{dr}\left(\frac{\upsilon_\vartheta}{r} \right) \right]^{n_i+1}. \tag{5.9}$$

To find the solution to this equation, develop the velocity υ_ϑ into a series

$$\upsilon_\vartheta = \upsilon_0 + \beta\upsilon_1 + \beta^2\upsilon_2 + \ldots, \tag{5.10}$$

where

$$\beta = \frac{\mu_i}{\mu_0}. \tag{5.11}$$

Putting now υ_ϑ from Eq.(5.10) into Eq.(5.9) and retaining only the two first terms, we will obtain for the same powers of β the following equations:

– for β^0

$$\frac{C_1}{\mu_0 r^2} = r\frac{d}{dr}\left(\frac{\upsilon_\vartheta}{r}\right), \tag{5.12}$$

– for β^1

$$r\frac{d}{dr}\left(\frac{\upsilon_1}{r}\right) = \left[-r\frac{d}{dr}\left(-\frac{\upsilon_0}{r}\right)\right]^{n_i+1}. \tag{5.13}$$

Upon solving these equations, we have

$$\upsilon_0 = C_2^{(0)}r - \frac{C_1}{2\mu_0 r} \tag{5.14}$$

and

$$\upsilon_1 = C_2^{(1)}r - \frac{\left(-C_1\right)^{n_i+1}}{\mu_0^{n_i+1}\left(2n_i+2\right)r^{2n_i+1}}. \tag{5.15}$$

Finally, according to Eq.(5.10), we have

$$\upsilon_\vartheta = C_2 r + \frac{\left(-C_1\right)}{2\mu_0 r} - \frac{\mu_i}{\mu_0}\frac{\left(-C_1\right)^{n_i+1}}{\mu_0^{n_i+1}\left(2n_i+2\right)r^{2n_i+1}}. \tag{5.16}$$

Note that on the basis of the second boundary condition (5.7) we have

$$C_2 = -\frac{\left(-C_1\right)}{2\mu_0 R_o^2} + \frac{k_i}{\mu_0}\frac{\left(-C_1\right)^{n_i+1}}{\mu_0^{n_i+1}\left(2n_i+2\right)R_o^{2n_i+2}} \tag{5.17}$$

and from the first boundary condition (5.6), we have

$$\omega = \frac{(-C_1)}{2\mu_0}\left(\frac{1}{R_i^2}-\frac{1}{R_o^2}\right)+\frac{\mu_i}{\mu_0}\frac{(-C_1)^{n_i+1}}{(2n_i+2)}\left(\frac{1}{R_i^{2n_i+2}}-\frac{1}{R_o^{2n_i+2}}\right). \tag{5.18}$$

The angular velocity $\omega(r)=\dfrac{\upsilon_\vartheta}{r}$ at any position r is expressed as

$$\omega(r) = C_2 + \frac{(-C_1)}{2\mu_0 r^2}+\frac{\mu_i}{\mu_0}\frac{(-C_1)^{n_i+1}}{\mu_0^{n_i+1}(2n_i+2)r^{2n_i+2}}, \tag{5.19}$$

then

$$\omega-\omega(r) = \frac{(-C_1)}{2\mu_0 r^2}\left(\frac{1}{\beta^2}-\frac{1}{\tilde{r}^2}\right)-\frac{\mu_i}{\mu_0}\frac{(-C_1)^{n_i+1}}{\mu_0^{n_i+1}(2n_i+2)R_o^{2n_i+2}}\left(\frac{1}{\beta^{2n_i+2}}-\frac{1}{\tilde{r}^{2n_i+2}}\right), \tag{5.20}$$

where

$$\beta=\frac{R_i}{R_o}, \qquad \tilde{r}=\frac{r}{R_o}. \tag{5.21}$$

The unit torque acting on the cylindrical surface of radius r is equal to

$$T = 2\pi r^2 \Lambda_{r\vartheta} = 2\pi C_1 . \tag{5.22}$$

Denoting, respectively, by T_s the torque acting on the inner cylinder to maintain its motion and by T_r the anti-torque applied to the outer cylinder to maintain its rest we have

$$T = \begin{cases} T_r = -2\pi C_1, \\ T_s = 2\pi C_1, \end{cases} \tag{5.23}$$

$$\omega-\omega(r) = \frac{1}{2\mu_0}\left(\frac{T}{2\pi R_o^2}\right)\left[\left(\frac{1}{\beta^2}-\frac{1}{\tilde{r}^2}\right)+\right.$$
$$\left.-\frac{\mu_i}{\mu_0}\left(\frac{T}{2\pi R_o^2}\right)^{n_i}\frac{1}{\mu_0^{n_i}(n_i+1)}\left(\frac{1}{\beta^{2n_i+2}}-\frac{1}{\tilde{r}^{2n_i+2}}\right)\right] \tag{5.24}$$

the formula which can be used for determining the material constants in the Sisko model from measurements of torque and angular velocity in a coaxial annular viscosimeter [21].

5. Conclusions

Simple flows of generalized second grade pseudoplastic fluids based on Sisko model or similar models may find many applications in a different branches of technology and industry. It can cite for example the theory of lubrication or petrochemical technology. Basing on the general equations of motions of the Sisko model of fluid it was presented equations of motion for thee flows, namely two Poiseuille flows in: plane channel, circular tube, and rotating Couette flow between two coaxial cylinders. The given solutions to Poiseuille flows may be used to modelling the flows in geological beds or porous layers whereas the Couette flow can serve to measuring the physical fluids parameters.

Nomenclature

A – square root from the second invariant of $\boldsymbol{A}_l$
$\boldsymbol{A}_n$ – n-th order kinematic tensor of Rivlin-Ericksen
$\boldsymbol{a}$ – acceleration vector
$\boldsymbol{D}$ – rate of deformation tensor
D_{ij} – component on tensor $\boldsymbol{D}$
e – Naperian logarithm base
$\boldsymbol{k}$ – third base vector in Cartesian coordinates
$\boldsymbol{L}$ – tensor of velocity gradient
M_i – viscosity function
n – exponential rheological parameter
p – pressure
$\boldsymbol{T}$ – shear stress tensor
T – pressure
t – time
$\boldsymbol{V}$ – velocity vector
υ_k – components of velocity vector
$\boldsymbol{I}$ – unit tensor
α_i, β_i – material coefficients for the second grade fluids
$\dot{\gamma}$ – shear strain rate
δ_{ij} – components of the unit tensor
Λ – extra stress tensor
μ – shear viscosity
ϑ – angular coordinate
μ_0, μ_∞ – limiting values of shear viscosity
ρ – fluid density
τ – shear stress
ω – angular viscosity

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