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# SCATTERING OF INTERNAL WAVES BY VERTICAL BARRIER IN A CHANNEL OF STRATIFIED FLUID

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The problem of two dimensional internal wave scattering by a vertical barrier in the form of a submerged plate, or a thin wall with a gap in an exponentially stratified fluid of uniform finite depth bounded by a rigid plane at the top, is considered in this paper. Assuming linear theory and the Boussinesq approximation, the problem is formulated in terms of the stream function. In the regions of the two sides of the vertical barrier, the scattered stream function is represented by appropriate eigen function expansions. By the use of appropriate conditions on the barrier and the gap, a dual series relation involving the unknown elements of the scattering matrix is produced. By defining a function with these unknown elements as its Fourier sine expansion series, it is found that this function satisfies a Carleman type integral equation on the barrier whose solution is immediate. The elements of the scattering matrix are then obtained analytically as well as numerically corresponding to any mode of the incident internal wave train for each barrier configuration. A comparison with earlier results available in the literature shows good agreement. To visualize the effect of the barrier on the fluid motion, the stream lines for an incident internal wave train at the lowest mode are plotted.

**Key words:** stratified fluid, internal wave, vertical barrier, stream function, scattering matrix, Boussinesq approximation, eigen function expansion.

## 1. Introduction

Within the framework of linearised theory, a train of internal waves is incident from infinity on a bottom standing thin vertical barrier present in a stratified fluid of uniform finite depth, was considered by Larsen (1969). He obtained the solution of the problem corresponding to incident internal wave of the lowest mode. The related problem of small horizontal oscillation of a barrier as a whole was investigated by Krutitsky (1988). Later, Korobkin (1990) studied the motion of a body taken as a dipole source in a weakly stratified fluid in the presence of a bottom obstacle modeled as a thin vertical bottom standing plane barrier by utilizing Larsen's (1969) solution. Recently, Dolai (2011), Dolai and Dolai (2013) studied the problems of internal wave scattering by a strip or elastic plate on the surface in a stratified fluid.

In this paper, we consider the problem of internal wave scattering by a vertical barrier in the form of a submerged plate or a wall with a submerged gap in an exponentially stratified fluid of uniform finite depth bounded above by a rigid plane. Due to the presence of the barrier in the stratified fluid, the incident waves (described by a stream function) are reflected back by the barrier with various modes and transmitted through the gap also with various modes. The reflected and transmitted internal waves are described by a scattering stream function which satisfies a boundary value problem in the fluid region. This stream function is expressed on both sides of the barrier by appropriate eigen function expansions involving the elements of the scattering matrix. By the use of appropriate conditions on the barrier and the gap, a dual series relation involving the elements of the scattering matrix is obtained. By defining a function on the barrier involving the elements of the scattering matrix as the coefficients of its Fourier series, the dual series relation reduces to a Carleman integral equation in a single integral for the submerged plate problem or in a double integral for the problem of a wall with a submerged gap. The solution of this integral equation for each case is obtained by standard technique. Using the solution of the appropriate integral equation, the elements of the scattering matrix and hence the stream function describing the resulting motion in the fluid for each case are obtained in principle, and for the lowest mode of the incoming internal wave train, these are evaluated explicitly. For both the barrier configurations, the stream lines are plotted graphically to visualize the effect of the barrier on the incoming internal wave train.

## 2. Formulation of the problem

A train of time-harmonic progressive internal waves is propagating along the positive $x'$-direction in a channel of a stratified fluid, where we choose a rectangular Cartesian co-ordinate system $x', y', z'$ in which the origin is taken as a point on the bottom, the $x'z'$-plane as the lower rigid boundary, the $y'$-axis vertically upwards and $y' = H$ as the upper rigid boundary.

For small motion in the fluid, let $u(x', y', t')$ and $v(x', y', t')$ denote the velocity components along $x', y'$ directions respectively. In the case of a weakly stratified fluid, the density and pressure denoted by $\rho_1(x', y', t')$ and $p_1(x', y', t')$ are given by

$$\rho_1(x', y', t') = \rho_0(y') + \rho(x', y', t'),$$
$$p_1(x', y', t') = g\int_{y'}^{H} \rho_0(y')\,dy' + p(x', y', t') \tag{2.1}$$

where $t'$ is the time, $\rho, p$ are perturbed density and pressure, respectively, and

$$\rho_0(y') = \rho_0(0)\exp\left(-\frac{y'}{L}\right). \tag{2.2}$$

Here, $g$ is the acceleration due to gravity, $\rho_0(0)$ is the density at the bottom and $L$ is the linear dimension characterizing stratification. Again, the continuity equation is

$$\frac{\partial u}{\partial x'} + \frac{\partial v}{\partial y'} = 0, \qquad \{2.3)$$

so that

$$u = -\frac{\partial \Psi}{\partial y'}, \qquad v = \frac{\partial \Psi}{\partial x'} \tag{2.4}$$

where $\Psi(x', y', t')$ is the stream function describing the motion in the fluid region. We assume that $u, v, p, \rho$ are small quantities so that their products and higher order derivatives can be neglected.

Assuming linear theory, the incompressibility condition produces

$$\frac{\partial \rho}{\partial t'} - \frac{\partial \Psi}{\partial x'}\frac{d\rho_0}{dy'} = 0, \tag{2.5}$$

and the Euler dynamical equations become

$$\rho_0(y')\frac{\partial u}{\partial t'} = -\frac{\partial p}{\partial x'}\,,$$
$$\rho_0(y')\frac{\partial v}{\partial t'} = -\frac{\partial p}{\partial y'} - \rho g\,. \tag{2.6}$$

Now, eliminating $p$ from the coupled Eqs in (2.6) and using (2.5), we find

$$\rho_0(y')\frac{\partial}{\partial t'}\left(\frac{\partial^2\Psi}{\partial x'^2}+\frac{\partial^2\Psi}{\partial y'^2}\right) = -g\frac{\partial\rho}{\partial x'} - \frac{d\rho_0(y')}{dy'}\frac{\partial^2\Psi}{\partial y'\partial t'}\,. \tag{2.7}$$

Again, using the relation (2.2) and eliminating $\rho$ from Eqs (2.7) and (2.5), we obtain

$$\frac{\partial^2}{\partial t'^2}\left(\frac{\partial^2\Psi}{\partial x'^2}+\frac{\partial^2\Psi}{\partial y'^2}\right) - \frac{1}{L}\frac{\partial^3\Psi}{\partial t'^2\partial y'} + \frac{g}{L}\frac{\partial^2\Psi}{\partial x'^2} = 0\,. \tag{2.8}$$

Using the non-dimensional variables $x$, $y$, $t$ defined by

$$x = \frac{\pi}{H}x', \qquad y = \frac{\pi}{H}y', \qquad t = \left(\frac{g}{L}\right)^{1/2} t'\,,$$

and defining $\psi(x,y,t) = \Psi\left(\frac{H}{\pi}x, \frac{H}{\pi}y, \left(\frac{L}{g}\right)^{1/2} t\right)$ and assuming $\frac{H}{L} << 1$ (which is equivalent to the Boussinesq approximation for a weakly stratified fluid), we find that

$$\nabla^2\frac{\partial^2\psi}{\partial t^2} + \frac{\partial^2\psi}{\partial x^2} = 0, \qquad -\infty < x < \infty, \qquad 0 < y < \pi\,, \tag{2.9}$$

which is the basic partial differential equation satisfied by the stream function $\psi(x,y,t)$.

Since the fluid lies between two rigid planes $y = 0$ and $y = \pi$, the bottom and the top conditions are

$$\psi = 0 \qquad \text{on} \qquad y = 0, \pi\,. \tag{2.10}$$

For time-harmonic motion with dimensionless frequency $\omega$, the time-dependence in $\psi(x,y,t)$ can be taken as $\psi(x,y,t) = \mathrm{Re}\left\{\phi^{\text{total}}(x,y)\exp(-i\omega t)\right\}$ where $\phi^{\text{total}}(x,y)$ now satisfies

$$\omega^2\nabla^2\phi^{\text{total}} = \frac{\partial^2\phi^{\text{total}}}{\partial x^2}, \qquad -\infty < x < \infty, \qquad 0 < y < \pi\,, \tag{2.11}$$

$$\phi^{\text{total}} = 0 \qquad \text{on} \qquad y = 0, \pi\,. \tag{2.12}$$

In the absence of the barrier, progressive wave solutions for $\phi(x,y)$ are

$$\exp\left\{\pm i\frac{k\omega x}{\sqrt{1-\omega^2}}\right\}\sin ky$$

where $k$ is the mode, and these exist only when $0<\omega<1$.

Let a train of progressive internal waves represented by the stream function $\mathrm{Re}\left\{\phi^{\mathrm{inc}}(x,y)\exp(-i\omega t)\right\}$ with

$$\phi^{\mathrm{inc}}(x,y)=\exp\left\{i\frac{k\omega x}{\sqrt{1-\omega^2}}\right\}\sin ky\,, \tag{2.13}$$

be propagating along the positive $x$-direction, where $k$ is the mode of the internal wave.

In the absence of a barrier, the train of internal wave of mode $k$ will propagate in the fluid without any distortion. However, due to the presence of the barrier the incident internal wave train will be scattered by the barrier. Let $\phi(x,y)$ denote the stream function of the scattered wave such that $\phi^{\mathrm{total}}=\phi^{\mathrm{inc}}+\phi$. Then $\phi(x,y)$ satisfies (2.11) and (2.12) together with

$$\phi=-\sin ky \qquad \text{on} \qquad x=0, \qquad y\in L \tag{2.14}$$

where $L=L_j\,(j=1,2)$. Here $L_1=(\alpha,\beta)$ corresponds to a submerged plate and $L_2=(0,\alpha)\cup(\beta,\pi)$ corresponds to a wall with a submerged gap.

Also,

$$\phi(x,y) \qquad \text{behaves as outgoing waves as} \qquad |x|\to\infty\,. \tag{2.15}$$

Again, as the scattered stream function $\phi(x,y)$ is symmetric about the barrier line (x=0) we have

$$\phi(x,y)=\phi(-x,y)\,. \tag{2.16}$$

Since $\frac{\partial\phi(+0,y)}{\partial x},\frac{\partial\phi(-0,y)}{\partial x}$ exist for all $y$, but $\frac{\partial\phi(0,y)}{\partial x}$ exists only across the gap, and as such because of Eq.(2.16), $\phi(x,y)$ must satisfy

$$\frac{\partial\phi(0,y)}{\partial x}=\frac{\partial\phi(\pm 0,y)}{\partial x}=0 \quad \text{for} \quad y\in\bar{L} \tag{2.17}$$

where $\qquad \bar{L}=(0,\pi)-L.$

## 3. Method of solution

The solution $\phi(x,y)$ satisfying Eq.(2.11) and the conditions (2.12), (2.16) and (2.15) are given by

$$\phi(x,y)=\sum_{m=1}^{\infty} p_m^k \exp\left\{i\frac{m\omega}{\sqrt{1-\omega^2}}|x|\right\}\sin my \tag{3.1}$$

where the unknown constants $p_m^k\ (m=1,2,3,\dots)$ may be interpreted as the transmission or reflection co-efficient for the $m^{\text{th}}$ transmitted $(m\neq k)$ or reflected mode corresponding to the $k^{\text{th}}$ incident mode. For $m=k, 1+p_k^k$ is the transmission co-efficient for the $k^{\text{th}}$ transmitted mode. We call $p_m^k$ an element of the scattering matrix for $m\neq k$.

The following dual series relation for the elements $p_m^k$ are obtained by using the conditions (2.17) and (2.14) when applied to Eq.(3.1)

$$\sum_{m=1}^{\infty} m\, p_m^k \sin my = 0 \qquad \text{for} \qquad y\in\bar{L}, \tag{3.2}$$

$$\sum_{m=1}^{\infty} p_m^k \sin my = -\sin ky \qquad \text{for} \qquad y\in L\ . \tag{3.3}$$

To solve the dual series relations (3.2) and (3.3), we consider a function $h_k(y)$ involving the elements $p_m^k$ as the Fourier sine series defined by

$$h_k(y)=-\frac{1}{k}\sum_{m=1}^{\infty} m\, p_m^k \sin my \qquad \text{for} \qquad 0<y<\pi, \tag{3.4}$$

then $h_k(y)$ is proportional to $\dfrac{\partial\phi(\pm 0,y)}{\partial x}$, the vertical component of scattering velocity at $x=0$. Because of the condition (3.2),

$$h_k(y)=0 \quad \text{for} \quad y\in\bar{L}, \tag{3.5}$$

and $h_k(y)$ is allowed to have at most a square root singularity near the edges of the barrier or plate.

The elements $p_m^k$ are related to $h_k(y)$ by

$$p_m^k=-\frac{2}{\pi}\frac{k}{m}\int_L h_k(y)\sin my\, dy\ . \tag{3.6}$$

Using Eq.(3.6) in Eq.(3.3) we find that $h_k(y)$ satisfies the integral equation

$$\frac{1}{\pi}\int_L h_k(u)\left\{\sum_{m=1}^{\infty}\frac{2\sin mu \sin my}{m}\right\}du=\frac{\sin ky}{k}, \qquad y\in L,$$

which is equivalent to

$$\frac{1}{\pi}\int_L h_k(u)\ln\left\{\frac{\sin\left(\frac{y+u}{2}\right)}{\sin\left|\frac{y-u}{2}\right|}\right\}du=\frac{\sin ky}{k}, \qquad y\in L. \tag{3.7}$$

This is a Carleman type integral equation and its solution can be obtained as follows.

Differentiation of both sides with respect to $y$ produces a first kind singular integral equation with a Cauchy type kernel in the form

$$\frac{1}{\pi}\int_{L'}\frac{G_k(t)}{t-s}dt=-\cos\left(k\cos^{-1}s\right), \qquad s\in L' \tag{3.8}$$

where the integral is in the sense of Cauchy principal value $G_k(t)=h_k\left(\cos^{-1}t\right), t=\cos u, \quad s=\cos y$ and $L'$ is the image of $L$. For $L=L_1=(\alpha,\beta)$, $L'=L_1^{/}=(b,a)$ with $a=\cos\alpha$, $b=\cos\beta\,(b<a)$ while for $L=L_2=(0,\alpha)\cup(\beta,\pi)$, $L'=L_2^{/}=(-1,b)\cup(a,1)$

We now deal with the cases $L=L_1$ and $L=L_2$, respectively.

**Case (a):** $L=L_1=(\alpha,\beta)$

This corresponds to a submerged plate. In this case $h_k(u)$ have square root singularities near $u=\alpha$ and $u=\beta$, the two edges of the submerged plate. Thus the solution of the integral Eq.(3.8) for $L'=L_1^{/}=(b,a)$ with the requirement that

$$G_k(t)=\begin{cases}\mathrm{O}\left\{(a-t)^{-1/2}\right\} & \text{as} \quad t\to a-0,\\ \mathrm{O}\left\{(t-b)^{-1/2}\right\} & \text{as} \quad t\to b+0\end{cases} \tag{3.9}$$

is given by (cf. Mikhlin (1964), Gakhov (1966), Cooke (1970))

$$G_k(t)=\frac{1}{\pi}\frac{1}{\{(t-b)(a-t)\}^{1/2}}\left\{\int_b^a\frac{\{(s-b)(a-s)\}^{1/2}}{s-t}\cos\left(k\cos^{-1}s\right)ds+C\right\}, \qquad b<t<a$$

where $C$ is an arbitrary constant and the integral is in the sense of Cauchy principal value. Thus

$$h_k(u) = G_k(\cos u) = \frac{1}{\pi}\frac{1}{\{(\cos\alpha - \cos u)(\cos u - \cos\beta)\}^{1/2}} \times$$
$$\times\left[\int_\alpha^\beta \{(\cos\alpha - \cos y)(\cos y - \cos\beta)\}^{1/2}\frac{\cos ky}{\cos y - \cos u}dy + C\right]. \tag{3.10}$$

The constant $C$ can be evaluated by substituting the solution $h_k(u)$ given by Eq.(3.10) in the original integral Eq.(3.7) for $L = L_1 = (\alpha, \beta)$, and then evaluating the various integrals for any $y \in (\alpha, \beta)$. However, these integrals cannot be obtained analytically, but can be evaluated numerically for various values of $y \in (\alpha, \beta)$ by Gauss quadrature taking care of the singular points and integer values of $k$. The following table gives $C$ numerically for $\alpha = 1, \beta = 2, k = 1,2,3$ and various values of $y_0$ in $(\alpha, \beta)$.

Table 1. Values of $C$.

| $y_0$ | $k = 1$ | $k = 2$ | $k = 3$ |
|---|---|---|---|
| *1.3* | *1.353287* | *0.061550* | *-0.267210* |
| *1.5* | *1.353327* | *0.061484* | *-0.267138* |
| *1.7* | *1.353302* | *0.061437* | *-0.267208* |
| *1.9* | *1.353098* | *0.061594* | *-0.267473* |

Thus $C$ is obtained numerically with an accuracy of three decimal places. It may be noted that $C$ is independent of $y$, and Tab.1 also demonstrates this fact.

The scattered elements $p_m^k$ are now obtained in principle by using Eq.(3.6) for $L = (\alpha, \beta)$ and $h_k(u)$ given by Eq.(3.10) after obtaining the constant $C$ as explained above. The scattered stream function $\phi(x, y)$ is then obtained by using the relation (3.1).

**Case (b):** $L = L_2 = (0, \alpha) \cup (\beta, \pi)$

This corresponds to a vertical wall with a submerged gap between $y = \alpha$ to $y = \beta$. Here $L' = L_2^/ = (-1, b) \cup (a, 1)$ is a double interval and $G_k(u)$ has square root singularities near $u = b$, $u = a$ and is zero near the end points $u = -1$, $1$ corresponding to the vicinity of the two ends of the wall. The appropriate solution to the integral Eq.(3.7) in this case satisfying the conditions at $t = \pm 1, b, a$ is given by (cf. Gakhov (1966))

$$G_k(t) = \frac{1}{\pi}\frac{\sqrt{1-t^2}}{\{(t-a)(t-b)\}^{1/2}}\int_{L_2^/}\frac{\{(s-a)(s-b)\}^{1/2}}{\sqrt{1-s^2}}\frac{\cos\left(k\cos^{-1}s\right)}{s-t}ds, \qquad t \in L_2^/. \tag{3.11}$$

Thus $h_k(u)$ in this case is given by

$$h_k(u) = \frac{1}{\pi} \frac{\sin u}{\{(\cos u - a)(\cos u - b)\}^{1/2}} \int_{L_2} \frac{\{(\cos y - a)(\cos y - b)\}^{1/2}}{\cos y - \cos u} \cos ky\, dy, \quad u \in L_2. \tag{3.12}$$

Now the scattering co-efficients $p_m^k$ are obtained from the relation given by Eq.(3.6) and hence the scattering stream function $\phi(x, y)$ is obtained by using the relation (3.1). It may be noted that if we make $\beta \to \pi$, then the upper part of the wall becomes non-existent and it assumes the form of a bottom standing barrier considered by Larsen (1969). The expression for $h_k(u)$ in Eq.(3.12) then becomes

$$h_k(u) = \frac{2}{\pi} \frac{\sin\frac{u}{2}}{\sqrt{\cos u - \cos\alpha}} \int_0^{\alpha} \frac{\sqrt{\cos y - \cos\alpha}}{\cos y - \cos u} \cos\frac{y}{2} \cos ky\, dy, \qquad 0 < u < \alpha.$$

For $k = 1$, this becomes

$$h_1(u) = \frac{2}{\pi} \frac{\sqrt{2}\sin\frac{u}{2}}{\sqrt{\cos u - \cos\alpha}} \left\{ \cos u + \frac{1 - \cos\alpha}{2} \right\}, \qquad 0 < u < \alpha,$$

which coincides with Larsen's (1969) result for a bottom standing barrier. The scattering co-efficients for $k = 1$ can be obtained explicitly in terms of Legendre polynomials and these are also given in Larsen's (1969) paper. For $\alpha = \pi/2$, the co-efficients $p_k^k\left(k = 1, 2, 3, \ldots\right)$ have been obtained analytically by Korobkin (1990). The scattering co-efficients $p_k^k\left(k = 1, 2, 3\right)$ are also obtained here directly as a limiting process from the plate problem or the wall with a gap problem by a numerical procedure. These coincide with the numerical values of $p_k^k$ given by Korobkin (1990) obtained analytically. This is discussed in some detail.

Again, if we make $\alpha \to 0$, then the lower part of the wall becomes non-existent and the wall assumes the form of a top-piercing barrier. In this case the expression for $h_k(u)$ in Eq.(3.12) becomes

$$h_k(u) = \frac{2}{\pi} \frac{\sin\frac{u}{2}}{\sqrt{\cos\beta - \cos u}} \int_{\beta}^{\pi} \frac{\sqrt{\cos\beta - \cos y}}{\cos y - \cos u} \sin\frac{y}{2} \cos ky\, dy, \qquad \beta < u < \pi.$$

For $k = 1$, the scattering co-efficients can be obtained explicitly in terms of Legendre polynomials.

## 4. Discussion

### 4.1. Scattering co-efficients

(a) *Submerged plate*: In this case $L = L_1 = \left(\alpha, \beta\right)$.

For a submerged plate, we have chosen $\alpha = 1,\ \beta = 2$ so that the vertical length (non-dimensional) of the plate is $1$. A representative set of the values of the scattering co-efficients $p_m^k\left(m = 1,2,3,\ k = 1,2,3\right)$ are given in the following table

Table 2. $p_m^k$ (for a submerged plate barrier).

| m / k | 1 | 2 | 3 |
|---|---|---|---|
| 1 | -0.857988 | -0.039243 | 0.171077 |
| 2 | -0.078591 | -0.402869 | -0.043459 |
| 3 | 0.515209 | -0.065256 | -0.236441 |

To check the validity of our numerical scheme, we have made $\alpha \to 0$ and $\beta = \pi / 2$ so that the plate assumes the form of a bottom-standing barrier. In this case the following table depicts $p_m^k \left( m = 1,2,3; k = 1,2,3 \right)$ for $\alpha = 0, \beta = \pi / 2$, computed from the results of the submerged plate barrier.

Table 3. $p_m^k$ (for a bottom-standing barrier obtained as a limiting case of plate).

| m / k | 1 | 2 | 3 |
|---|---|---|---|
| 1 | -0.747489 | -0.250445 | 0.061265 |
| 2 | -0.502668 | -0.437156 | -0.248718 |
| 3 | 0.183744 | -0.374258 | -0.560472 |

The values of $p_1^1, p_2^2$ and $p_3^3$ have been respectively given explicitly by Korobkin (1990) as *-3/4*, *-7/16*, *-9/16* i.e., *-0.75*, *-0.4375*, *-0.5625* who calculated these directly from the bottom standing barrier problem, and these coincide with our results upto two decimal places. This shows the correctness of the numerical scheme used here.

The scattering co-efficients for a top-piercing barrier can also be obtained from the results of the plate problem by making $\beta \to \pi$. In this case, the following table depicts $p_m^k \left( m = 1,2,3; k = 1,2,3 \right)$ for $\alpha = \pi / 2$.

Table 4. $p_m^k$ (for a top-piercing barrier obtained as a limiting case of plate).

| m / k | 1 | 2 | 3 |
|---|---|---|---|
| 1 | -0.747309 | 0.250355 | 0.061296 |
| 2 | 0.502932 | -0.437289 | 0.248763 |
| 3 | 0.183478 | 0.374392 | -0.560517 |

It may be noted that two geometrical configurations for which Tabs 3 and 4 are depicted are complementary to each other. If $p_m^{(1)k}$ and $p_m^{(2)k}$ correspond to bottom-standing and top-piercing barriers whose vertical lengths are equal to half the channel depth, then it can be shown that

$$p_m^{(1)k} = (-1)^{m+k}\, p_m^{(2)k}$$

This is also reflected in the Tabs 3 and 4.

(b) *Wall with a submerged gap*: In this case $L = L_2 = (0,\alpha) \cup (\beta,\pi)$

For a wall with a gap, we have chosen $\alpha = 1,\ \beta = 2$ so that the length (non-dimensional) of the vertical expanse of the gap is unity. A representative set of the values of the scattering co-efficients $p_m^k\ (m = 1,2,3;\ k = 1,2,3)$ are given in Tab.5.

Table 5. $p_m^k$ (for a wall with a gap with $\alpha = 1, \beta = 2$).

| m / k | 1 | 2 | 3 |
|---|---|---|---|
| 1 | -0.933026 | -0.092683 | -0.214593 |
| 2 | -0.033524 | -1.050250 | -0.047305 |
| 3 | -0.618495 | -0.366265 | -0.645788 |

Again, to check the validity of the numerical scheme followed here, we have made $\beta \to \pi$ and $\alpha = \pi/2$ so that the wall assumes the form of a bottom standing barrier. Table 6 depicts $p_m^k\ (m = 1,2,3;\ k = 1,2,3)$ for this case.

Table 6. $p_m^k$ (for a bottom-standing barrier obtained as a limiting case of a wall with a gap).

| m / k | 1 | 2 | 3 |
|---|---|---|---|
| 1 | -0.746568 | -0.250057 | 0.061365 |
| 2 | -0.503799 | -0.437415 | -0.248697 |
| 3 | 0.182792 | -0.374831 | -0.560893 |

A comparison of Tab.6 with Tab.3 shows that the elements in Tab.6 coincide with corresponding elements in Tab.3 upto 2 to 3 decimal places in most cases. Also, the diagonal elements agree with Korobkin's (1990) explicit values.

Again, if we make $\alpha \to 0$, then the wall assumes the form of a top-piercing barrier. In this case Tab.7 depicts $p_m^k\ (m = 1,2,3;\ k = 1,2,3)$ for $\beta = \pi/2$.

Table 7. $p_m^k$ (for a top-piercing barrier obtained as a limiting case of a wall with a gap) .

| *m* / *k* | *1* | *2* | *3* |
|---|---|---|---|
| *1* | *-0.746568* | *0.250057* | *0.061265* |
| *2* | *0.503799* | *-0.437415* | *0.248697* |
| *3* | *0.182791* | *0.374831* | *-0.560893* |

The entries in Tab.7 coincide with the entries in Tab.4 almost 2 to 3 decimal places. It may be noted that while the entries in Tab.7 are calculated as a limiting process from the problem of a wall with a gap and the entries in Tab.4 are calculated as a limiting process from the plate problem. This again confirms the correctness of the numerical method utilized here.

### 4.2. Stream function

To visualize the effect of the vertical barrier on the wave motion, the incident internal wave field of the lowest mode (i.e., $k = 1$) is chosen for simplicity. In this case the stream function describing the motion in the fluid is obtained as

$$\psi^{\text{total}}(x,y,t) = \text{Re}\left\{\exp\left[i\omega\left\{\frac{x}{\sqrt{1-\omega^2}} - t\right\}\right]\sin y + \sum_{m=1}^{\infty} p_m^1 \sin my \exp\left[i\omega\left\{\frac{m|x|}{\sqrt{1-\omega^2}} - t\right\}\right]\right\}$$

where the scattering co-efficients $p_m^1\ (m = 1,2,\ldots)$ can be obtained at least numerically. For the two configurations of the barrier, as well as the bottom-standing and top-piercing barriers as limiting cases of the plate as well as the wall with a gap, these co-efficients have been obtained numerically. In the absence of the barrier, the stream lines corresponding to the incident internal wave with mode one are depicted in Fig.1, taking $\omega = 0.6$ and $t = 5$ in the vicinity of $x = 0$ .

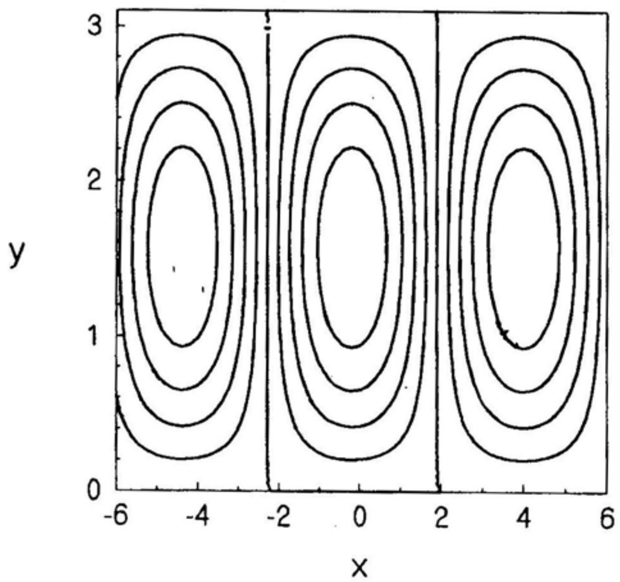


Fig.1. Stream lines in the absence of a barrier for *k=1*, $\omega = 0.6$ , *t=5*.

When a barrier is introduced along the line *x=0*, the corresponding stream lines are depicted in Figs 2 to 5.

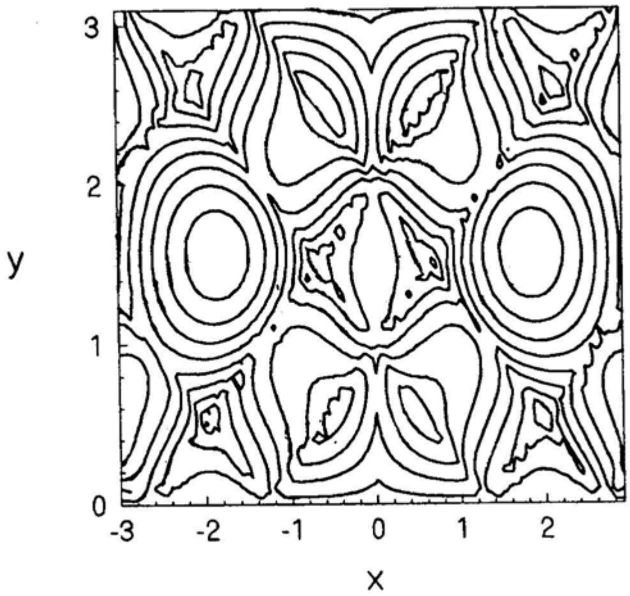


Fig.2. Stream lines for a submerged plate of unit length $(\beta = 2, \alpha = 1)$ with $k=1$, $\omega = 0.6$ , $t=5$.

Figure 2 shows the streamlines for a submerged plate of unit length $(\alpha = 1, \beta = 2)$. Figure 3 shows the same for a thin wall with a gap of unit length $(\alpha = 1, \beta = 2)$.

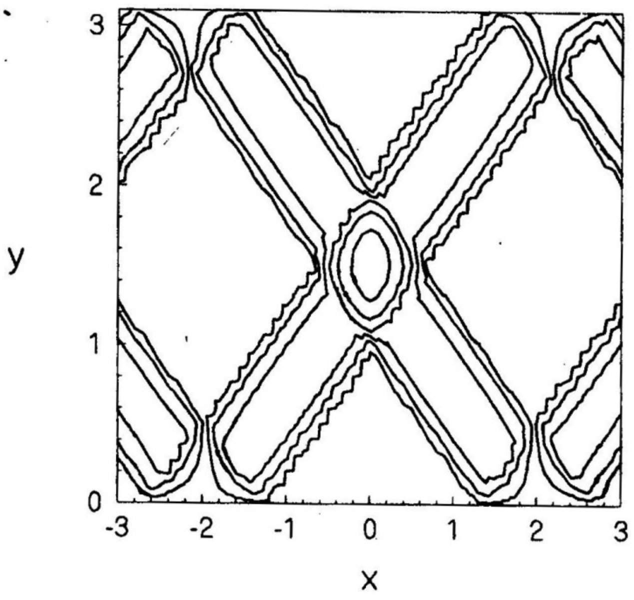


Fig.3. Stream lines for a thin wall with a gap of unit length $(\beta = 2, \alpha = 1)$ with $k=1$, $\omega = 0.6$ , $t=5$.

Figures 4 and 5 show the stream lines for a bottom-standing and top-piercing barrier. In all these figures $\omega = 0.6$ , $t=5$ have been taken.

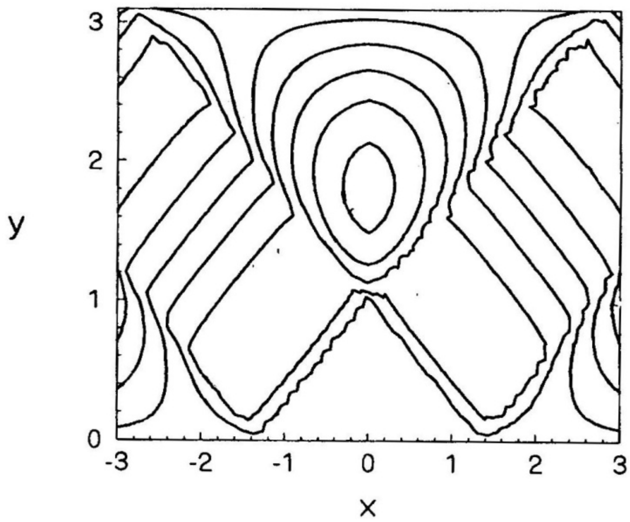


Fig.4. Stream lines for a bottom standing barrier $(\beta \to \pi, \alpha = 1)$ with $k=1$, $\omega = 0.6$ , $t=5$.

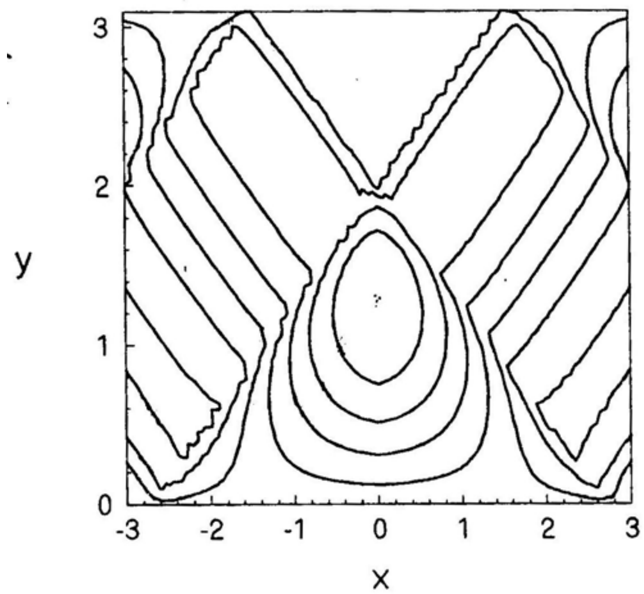


Fig.5. Stream lines for a top-piercing barrier $(\alpha \to 0, \beta = 2)$ with $k=1$, $\omega = 0.6$ , $t=5$.

The results for the last two cases have been obtained by making $\beta \to \pi$ (with $\alpha = 1$) and $\alpha \to 0$ (with $\beta = 2$) in the results of the wall with a gap configuration. The same results are also obtained by making $\alpha \to 0$ (with $\beta = 1$) and $\beta \to \pi$ (with $\alpha = 2$) in the results of the submerged plate configuration. In all the cases, it is observed that some stream lines abruptly change their direction, and the points where the changes in the direction of stream lines occur roughly lie on straight lines which intersect near the edges of the barriers. Different patterns in the stream line contours are formed for each of the barriers.

## 5. Conclusion

Scattering of internal waves by a thin vertical barrier in the form of a submerged plate or wall with a gap submerged in an exponentially stratified fluid of uniform finite depth under a rigid plane is studied here under the assumption of linear theory and Boussinesque approximation. The problem is formulated in terms of the stream function describing the motion in the fluid. The elements of the scattering matrix and the stream function are obtained through the solution of a Carleman integral equation for any mode of the incident internal wave. The scattering coefficients are also obtained numerically and some results are compared with the results available in the literature. Good agreement is seen to have been achieved. For the lowest mode of the incident internal waves, the stream lines are drawn to visualize pictorially the effect of the barrier for its various configurations.

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## Nomenclature

$g$ – acceleration due to gravity
$H$ – vertical length of the channel
$k$ – mode of the internal wave
$p$ – fluid perturbed density
$p_1$ – fluid pressure at any point
$t'$ – time
$u, v$ – velocity components
$x', z'$ – horizontal distance
$y'$ – vertical distance
$\beta - \alpha$ – length of the plate or gap
$\rho$ – fluid perturbed density
$\rho_0$ – fluid density at the bottom
$\rho_1$ – fluid density at any point
$\phi$ – scattered stream function
$\Psi$ – stream function
$\omega$ – circular frequency

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