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Brief note

INELASTIC STABILITY ANALYSIS OF UNIAXIALLY COMPRESSED FLAT RECTANGULAR ISOTROPIC CCSS PLATE

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This study investigates the inelastic stability of a thin flat rectangular isotropic plate subjected to uniform uniaxial compressive loads using Taylor-Maclaurin series formulated deflection function. The plate has clamped and simply supported edges in both characteristic directions (CCSS boundary conditions). The governing equation is derived using a deformation plasticity theory and a work principle. Values of the plate buckling coefficient are calculated for aspect ratios from *0.1* to *2.0* at intervals of *0.1*. The results compared favourably with the elastic stability values and the percentage differences ranged from *-0.353*% to *-7.427*%. Therefore, the theoretical approach proposed in this study is recommended for the inelastic stability analysis of thin flat rectangular isotropic plates under uniform in-plane compression.

Key words: deflection function, plastic buckling, rectangular plate, Taylor's series, work principle

1. Introduction

Thin rectangular plate elements are commonly used in thin-walled engineering structures to transmit in-plane and/or lateral loads. A plate is classified as "thin" if the ratio of its thickness to its smaller characteristic dimension is less than 1/20 (Ugural, 1999). When a thin rectangular plate is subjected to in-plane compressive loads and the loads are gradually increased, the plate loses its stability and begins to buckle at a critical value of the compressive loads even in the absence of transverse loads. Buckling may occur in the elastic range where the plate material obeys Hooke's law and the buckling stress is less than the proportional limit of the plate material. On the other hand, inelastic buckling is characterized by nonlinear stress-strain relationship. In the inelastic range, the actual buckling load is always smaller than the elastic buckling load. Therefore, it is important to know the inelastic buckling characteristics in order to accurately predict the critical buckling loads in the inelastic range.

Many theoretical and experimental studies have been conducted in the past decades to obtain the critical buckling loads of plates in the inelastic range. Despite the fact that such studies have been carried out extensively, several aspects in the theory of inelastic plate buckling are still controversial primarily because of the difficulty in the proper representation of the stress-strain relationship (Szilard, 2004). The most commonly used plate plasticity theories are the deformation theory developed by Ilyushin (1947) and the incremental theory or flow theory developed by Handelman and Prager (1948). Even though the incremental theory has a strong mathematical basis, it predicts results which are unreasonably higher than experimental values when applied to inelastic stability problems of homogenous plates. The deformation theory, on the other hand, gives results which are in closer agreement with experiments. Some researchers have thus continued to use the deformation theory in solving inelastic stability problems of plate elements despite its weak theoretical formulation.

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In finding solutions to plate inelastic stability problems of various boundary conditions, investigators have used various methods (Stowell, 1948; Isakson and Pifko, 1969; Guran-Savadkuhi, 1981; Iyengar, 1988; Shen, 1990; Wang *et al.*, 2004; Wang *et al.*, 2005; Becque, 2010). Most of these solutions were obtained using the Fourier series or trigonometric series irrespective of the plasticity theory or analytical method applied. In using the trigonometric series in the energy approach, formulation of the deflection function for certain boundary conditions becomes very difficult (Ugural, 1999; Ventsel and Krauthammer, 2001). Because of these limitations in using trigonometric series, the Taylor series may be used. The application of the Taylor's series in plate stability analysis has not attracted much attention in literature. To the best of the researchers' knowledge, the Taylor series has not been used in the energy method to formulate the deflection function for the inelastic stability analysis of CCSS plates, and solutions to the inelastic stability problem of CCSS plates are unavailable in open literature. Therefore, the objective of the present investigation is to provide a solution to the inelastic stability problem of a CCSS thin rectangular isotropic plate using a work principle and Taylor's series formulated deflection function. The deformation theory of plasticity proposed by Stowell (1948) is used to analyze the inelastic stability behaviour.

2. Methodology

2.1. Formulation of the stability problem

Consider a homogenous rectangular flat isotropic plate and assume that the thickness of the plate in the z-direction is small compared with its length and width in the x- and y-directions respectively. The thin rectangular plate is subjected to uniform in-plane compressive loads along the longitudinal axis (x-direction). The plate is clamped along one longitudinal edge, clamped along one short edge, simply supported along one longitudinal edge and simply supported along one short edge as illustrated in Fig.1. In Fig.1, C represents a clamped edge while *S* represents a simply supported edge.

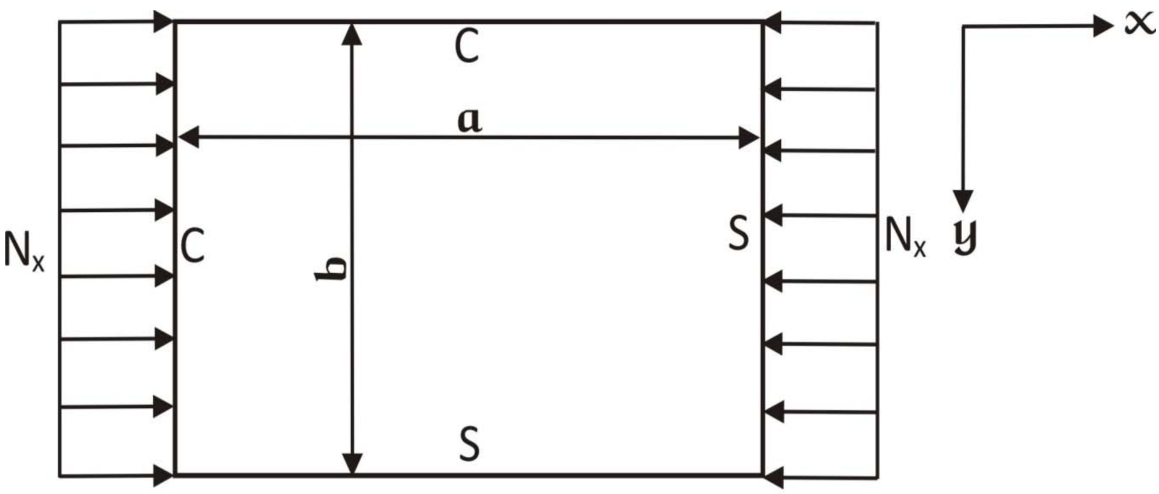

Fig.1. CCSS plate under uniaxial in-plane compression in Cartesian coordinates.

To facilitate the solution of the problem, the Cartesian coordinates are expressed in dimensionless parameters as

$$R = x / a; \qquad Q = y / b . \tag{2.1}$$

For the clamped edges, the deflection and rotation vanish while the deflection and moments are equal to zero along the simply supported edges. Thus, the boundary conditions of the CCSS plate are

$$w(R=0)=w'^{R}(R=0)=0\,, \tag{2.2}$$

$$w(R=1)=w''^{R}(R=1)=0\,, \tag{2.3}$$

$$w(Q=0)=w'^{Q}(Q=0)=0\,, \tag{2.4}$$

$$w(Q=1)=w''^{Q}(Q=1)=0 \tag{2.5}$$

where w is the deflection in the z-direction, w'^{R} and w'^{Q} are the first derivatives of the deflection and w''^{R} and w''^{Q} are the second derivatives of the deflection in the R-direction and Q-direction respectively.

2.2. Inelastic stability equation

Stowell (1948) expressed the equation governing the inelastic stability of a thin, flat, rectangular plate subjected to uniform axial compression in the x-direction as

$$\left(\frac{1}{4}+\frac{3}{4}\frac{T}{S}\right)\frac{\partial^4 w}{\partial x^4}+2\frac{\partial^4 w}{\partial x^2 \partial y^2}+\frac{\partial^4 w}{\partial y^4}=\frac{N_x}{\bar{D}}\frac{\partial^2 w}{\partial x^2} \tag{2.6}$$

where T is the tangent modulus, S is the secant modulus, $\bar{D}$ is the inelastic flexural rigidity and N_x is the buckling load.

Work is defined mathematically as the product of force and displacement produced by the force in the direction of the force. Ibearugbulem *et al.* (2013) solved bending problems of isotropic rectangular plates using direct integration and a work principle. They carried out direct integration of the governing differential equation of isotropic rectangular plates to obtain shape functions instead of assuming the shape function – a technique more commonly used in the conventional equilibrium and variational methods. They derived equations for deflections by using the principle of equilibrium of works performed by the load and the reaction (i.e., plate resistance). Eziefula (2013) applied this approach and transformed Eq.(2.6) using the principle of conservation of work in a static continuum. He multiplied the equation of force equilibrium by the deflection and integrated the resulting equation in a closed domain. Making N_x the subject of the formula, he obtained

$$N_x=\frac{\dfrac{\bar{D}p^2}{b^2}\displaystyle\int_0^1\!\!\int_0^1\left[\frac{1}{p^4}\left(\frac{1}{4}+\frac{3T}{4S}\right)\frac{H\partial^4 H}{\partial R^4}+\frac{2}{p^2}\frac{H\partial^4 H}{\partial R^2\partial Q^2}+\frac{H\partial^4 H}{\partial Q^4}\right]\partial R\partial Q}{\displaystyle\int_0^1\!\!\int_0^1 H\frac{\partial^2 H}{\partial R^2}\partial R\partial Q} \tag{2.7}$$

where

$$w=AH\,, \tag{2.8}$$

$$\bar{D}=Sh^3/9\,, \tag{2.9}$$

$$p=a/b\,. \tag{2.10}$$

In Eqs (2.7) to (2.10), A is the amplitude of the deflection function, H is the buckling curve expression, h is the thickness of the plate, a is the length of the plate, b is the width of the plate and p is the aspect ratio.

2.3. Taylor's series formulated deflection function

Ibearugbulem (2012) expanded the deflection function using Taylor's series and he assumed that the deflection function is differentiable and continuous. Truncating the infinite polynomial series at $m = n = 4$, he got

$$w=\sum_{m=0}^{4}\sum_{n=0}^{4}J_m K_n R^m Q^n \,. \tag{2.11}$$

For the CCSS boundary conditions, Eqs (2.2) to (2.5) are applied in Eq.(2.11). Substituting Eqs (2.2) and (2.4) into Eq.(2.11) gives

$$J_0=J_1=0; \qquad K_0=K_1=0\,.$$

Substituting Eq.(2.3) into Eq.(2.11) and solving the simultaneous equations yields

$$J_2=1.5J_4; \qquad J_3=-2.5J_4\,.$$

Similarly, substituting Eq.(2.5) into Eq.(2.11) and solving the resulting simultaneous equations gives

$$K_2=1.5K_4; \qquad K_3=-2.5K_4\,.$$

Substituting the values of J_0, J_1, J_2, J_3, J_4, K_0, K_1, K_2, K_3 and K_4 into Eq.(2.11) and solving gives the distinctive deflection function of the CCSS plate. This deflection function is expressed as

$$w=J_4K_4\left[\left(1.5R^2-2.5R^3+R^4\right)\left(1.5Q^2-2.5Q^3+Q^4\right)\right]. \tag{2.12}$$

From Eqs (2.8), (2.11) and (2.12), we have

$$A=J_4K_4\,, \tag{2.13}$$

$$H=\left(1.5R^2-2.5R^3+R^4\right)\left(1.5Q^2-2.5Q^3+Q^4\right). \tag{2.14}$$

2.4. Application of a work principle

Partial derivatives of Eq.(2.14) with respect to R, Q or both R and Q gave

$$H\frac{\partial^4 H}{\partial R^4}=24\left(1.5R^2-2.5R^3+R^4\right)\left(1.5Q^2-2.5Q^3+Q^4\right)^2 \,, \tag{2.15}$$

$$H\frac{\partial^4 H}{\partial Q^4}=24\left(1.5R^2-2.5R^3+R^4\right)^2\left(1.5Q^2-2.5Q^3+Q^4\right), \tag{2.16}$$

$$H\frac{\partial^4 H}{\partial R^2 \partial Q^2} = 9\left(1-5R+4R^2\right)\left(1-5Q+4Q^2\right)$$
$$\left(1.5R^2-2.5R^3+R^4\right)\left(1.5Q^2-2.5Q^3+Q^4\right), \tag{2.17}$$

$$H\frac{\partial^2 H}{\partial R^2} = 3\left(1-5R+4R^2\right)\left(1.5R^2-2.5R^3+R^4\right)\left(1.5Q^2-2.5Q^3+Q^4\right)^2 . \tag{2.18}$$

Equations (2.15) to (2.18) are expanded and integrated partially with respect to R and Q respectively in a closed domain. The results in five significant figures are

$$\int_0^1\int_0^1 H\frac{\partial^4 H}{\partial R^4}\partial R\partial Q = 0.013571 , \tag{2.19}$$

$$\int_0^1\int_0^1 H\frac{\partial^4 H}{\partial Q^4}\partial R\partial Q = 0.013571 , \tag{2.20}$$

$$\int_0^1\int_0^1 H\frac{\partial^4 H}{\partial R^2\partial Q^2}\partial R\partial Q = 0.0073470 , \tag{2.21}$$

$$\int_0^1\int_0^1 H\frac{\partial^2 H}{\partial R^2}\partial R\partial Q = 0.00064626 . \tag{2.22}$$

Substituting the values of the integrals in Eqs (2.19) to (2.22) into Eq.(2.11) gives

$$N_x = \left[\frac{\dfrac{0.013571}{p^2}\left(\dfrac{1}{4}+\dfrac{3T}{4S}\right)+0.014694+0.013571p^2}{0.00064626}\right]\frac{\overline{D}}{b^2} . \tag{2.23}$$

Generally, the plate buckling equation is often expressed as

$$N_x = \frac{k\pi^2\overline{D}}{b^2} \tag{2.24}$$

where k is the plate buckling coefficient. If Eq.(2.23) is written in the form of Eq.(2.24), then the plate buckling coefficient may be expressed as

$$k = \frac{2.12767}{p^2}\left(\frac{1}{4}+\frac{3T}{4S}\right)+2.30374+2.12767p^2 . \tag{2.25}$$

3. Results and discussion

The present investigation derived the inelastic critical load equation for the CCSS boundary conditions of the thin rectangular isotropic plate as

$$\left(N_x\right)_{cr}=\left[\frac{2.12767}{p^2}\left(\frac{1}{4}+\frac{3T}{4S}\right)+2.30374+2.12767\,p^2\right]\frac{\pi^2\bar{D}}{b^2}. \tag{3.1}$$

Ibearugbulem (2012) gave the critical load equation for the elastic stability of the same plate as

$$\left(N_x\right)_{cr}=\frac{D\pi^2}{b^2}\left(\frac{2.1278}{p^2}+2.1278\,p^2+2.303\right). \tag{3.2}$$

Both solutions in Eqs (3.1) and (3.2) are upper bound and approximate. Both solutions also employed the Taylor series formulated deflection function. The present investigation based on Stowell's method used $\bar{D}$ instead of D. The notation, D, is the elastic flexural rigidity and is expressed as

$$D=\frac{Eh^3}{12\left(1-\nu^2\right)} \tag{3.3}$$

where ν is the Poisson ratio and E is the modulus of elasticity. Stowell (1948) used a numerical value of 0.5 for the value of the Poisson ratio during the inelastic buckling and expressed $\bar{D}$ as shown in Eq.(2.9).

From Eq.(3.1), it may be noted that k is a function of p, T and S. A thorough knowledge of the stress-strain relationship of the plate material in the inelastic range is required in order to calculate the values of T and S. The factor T/S is always less than one in the inelastic buckling analysis. If its value is expressed as $T/S = 1$, then the value of the critical buckling load for the inelastic buckling will be equal to that of the elastic buckling. In the present investigation, numerical values of T/S equal to 0.6, 0.75 and 0.9 are used to calculate the values of the aspect ratios $0.1 \le p \le 2.0$ at intervals of 0.1. The values of k are shown in Tab.1.

From Tab.1, the average percentage difference between the solution from the present investigation for $T/S = 0.9$ and the solution for Ibearugbulem (2012) is - 3.040%. It may be observed that the percentage difference between the two solutions improves as the aspect ratio increases. However, solutions were unavailable in open literature for comparing the results of the present investigation and Ibearugbulem (2012) for the CCSS boundary conditions. Table 2 shows the minimum ($p = 0.1$) and maximum ($p = 2.0$) values of k for the inelastic stability ($T/S = 0.9$) and elastic stability of plates with CCSS, CCCC and CSSS boundary conditions, as well as the minimum ($p = 0.1$) and maximum ($p = 2.0$) percentage differences.

The data in Tab.2 indicates that the minimum and maximum percentage differences for the CCSS, CCCC and CSSS boundary conditions have similar numerical values. According to Ibearugbulem (2012), the elastic stability solutions for the CCCC and CSSS plates compared favourably with those of previous research studies. The average percentage difference for Ibearugbulem (2012) and Iyengar (1988) was 3.538% for $0.1 \le p \le 1.0$ at intervals of 0.1 for the CCCC boundary conditions. For the CSSS boundary conditions, the difference with Michelutti (1976) was 0.55% for $p = 0.79$ while the difference with Fok (1980) was 6.25% for $p = 1.0$ as cited in Ibearugbulem (2012). These differences are quite close to one another and are acceptable in statistics. Further research could be conducted to confirm the accuracy of the results presented in this paper.

Table 1. Values of *k* for uniaxially compressed CCSS plate.

p	*k* from Present Investigation			*k* from Ibearugbulem (2012)	α [*]
	T/S = 0.60	*T/S = 0.75*	*T/S = 0.90*		
0.1	151.262	175.198	199.134	215.104	–7.424
0.2	39.623	45.607	51.591	55.583	–7.182
0.3	19.044	21.703	24.363	26.137	–6.787
0.4	11.953	13.449	14.945	15.942	–6.254
0.5	8.793	9.751	10.708	11.346	–5.623
0.6	7.207	7.872	8.537	8.980	–4.933
0.7	6.386	6.874	7.363	7.688	–4.227
0.8	5.993	6.367	6.741	6.989	–3.548
0.9	5.866	6.161	6.457	6.653	–2.946
1.0	5.921	6.160	6.400	6.559	–2.424
1.1	6.109	6.307	6.505	6.636	–1.974
1.2	6.402	6.568	6.734	6.845	–1.621
1.3	6.781	6.922	7.064	7.158	–1.313
1.4	7.234	7.356	7.478	7.559	–1.072
1.5	7.753	7.859	7.966	8.036	–0.871
1.6	8.332	8.426	8.519	8.581	–0.723
1.7	8.968	9.051	9.134	9.189	–0.599
1.8	9.657	9.731	9.805	9.854	–0.497
1.9	10.397	10.463	10.530	10.574	–0.416
2.0	11.187	11.247	11.306	11.346	–0.353

[*] α means percentage difference between *k* from Present Investigation (*T/S = 0.90*) and Ibearugbulem (2012)

Table 2. Inelastic values (*T/S = 0.9*) and elastic values of *k* for selected boundary conditions.

Boundary Conditions	[a] Inelastic *k*		Elastic *k* (Ibearugbulem 2012)		Percentage Difference	
	p = 0.1	*p = 2.0*	*p = 0.1*	*p = 2.0*	*p = 0.1*	*p = 2.0*
CCSS	199.134	11.306	215.104	11.346	–7.424	–0.353
CCCC	396.099	20.439	427.970	20.512	–7.447	–0.366
CSSS	94.948	12.337	102.429	12.354	–7.304	–0.138

[a] Solutions for the inelastic plate buckling of the SCCC, CCCC and CSSS boundary conditions are found in this Present Investigation, Ibearugbulem *et al.* (2013b) and Eziefula *et al.* (2013) respectively.

4. Conclusions

This investigation was designed to analyze the inelastic stability of a thin flat rectangular isotropic plate subjected to uniaxial in-plane compression with clamped and simply supported edges in both characteristics dimensions. The deformation theory of plasticity based on Stowell's method and a work principle were used to derive the inelastic governing equation. The deflection function was formulated with the Taylor series. The results of the investigation were compared with the elastic buckling values and the results show strong consistency. The idea of the results is to provide an alternative means of estimating the deflection and buckling coefficient of thin flat rectangular isotropic plates as well as to provide a solution for the CCSS boundary conditions. The approach proposed in this investigation could be extended to the inelastic stability of thin rectangular isotropic plates of other boundary conditions. This approach is valid for continuous and isotropic plates subjected to proportional loading for a range of metal materials.

Nomenclature

A – amplitude of the deflection function
a – length of plate
b – width of plate
C – clamped edge
CCSS – rectangular plate clamped along one longitudinal edge, clamped along one short edge, simply supported along one longitudinal edge, and simply supported along one short edge
D – plate flexural rigidity in the elastic range
$\bar{D}$ – plate flexural rigidity in the plastic range
E – Young's modulus
H – buckling curve expression
h – thickness of plate
J, K – unknown constants in the polynomial series
k – plate buckling coefficient
m – number of half-waves of the buckling mode along the x-direction
N_x – uniaxial in-plane compressive load on x-plane
$N_{x,CR}$ – critical buckling load
n – number of half-waves of the buckling mode along the y-direction
p – aspect ratio
R, Q – non-dimensional axes of the x- and y-coordinates respectively
S – simply supported edge
S – secant modulus
T – tangent modulus
w – out-of-plane deflection
w'^R, w'^Q – first derivative of the deflection in the R- and Q-coordinates, respectively
w''^R, w''^Q – second derivative of the deflection in the R- and Q-coordinates, respectively
x,y – Cartesian coordinates in the horizontal and vertical direction, respectively
ν – Poisson ratio

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