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THE EFFECT OF RANDOMNESS ON THE STABILITY OF CAPILLARY GRAVITY WAVES IN THE PRESENCE OF AIR FLOWING OVER WATER

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A nonlinear spectral transport equation for the narrow band Gaussian random surface wave trains is derived from a fourth order nonlinear evolution equation, which is a good starting point for the study of nonlinear water waves. The effect of randomness on the stability of deep water capillary gravity waves in the presence of air flowing over water is investigated. The stability is then considered for an initial homogenous wave spectrum having a simple normal form to small oblique long wave length perturbations for a range of spectral widths. An expression for the growth rate of instability is obtained; in which a higher order contribution comes from the fourth order term in the evolution equation, which is responsible for wave induced mean flow. This higher order contribution produces a decrease in the growth rate. The growth rate of instability is found to decrease with the increase of spectral width and the instability disappears if the spectral width increases beyond a certain critical value, which is not influenced by the fourth order term in the evolution equation.

Key words: capillary gravity waves, randomness, evolution equation, instability.

AMS Mathematics subject classification: 76B15, 76E17.

1. Introduction

An analysis of the evolution of nonlinear surface water waves has been considered to treat the problem either from the deterministic point of view, with emphasis on the properties and stability of nonlinear water waves or from the random point of view, with emphasis on wave-wave energy transfer within a broad spectrum due to weak nonlinear couplings in a nearly homogenous random ocean. In the study of nonlinear wave-wave interactions near the gravity wave spectrum, Longuet-Higgins (1975) made the analysis by joining the two wave view points together.

Starting from Davey-Stewartson (1974) third order nonlinear evolution equation for deep water surface gravity waves, the effect of randomness on the stability of two dimensional surface gravity waves was studied by Alber (1978). Alber then obtained a spectral transport equation which is used to study the stability of an initially homogeneous wave spectrum. A Gaussian spectrum was considered by Alber (1978), while a Lorentz shape of the spectrum was considered by Crawford *et al*. (1980). In both the papers, it was shown that there is instability if the effective bandwidth is less than a critical value and the effect of randomness is to reduce the growth rate of instability. The deterministic results of Benjamin and Feir (1967) were recovered by taking the effective bandwidth to zero.

Dysthe (1979) showed that a fourth order nonlinear evolution equation, which is one order higher than the lowest order evolution equation, is a good starting point for study the nonlinear surface waves in deep water for waves of wave steepness up to *0.25*. The stability analysis made from the fourth order nonlinear evolution equation gives result which is consistent with the exact result of Longuet-Higgins

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(1978a; b) and with the experimental results of Benjamin and Feir (1967) for wave steepness up to *0.25*. The dominant new effect that comes in the fourth order is the influence of wave induced mean flow and this produces a significant deviation in the stability character. Fourth order nonlinear evolution equation for deep water surface gravity waves including different effects were derived and stability analysis was considered by several authors (Stiassnie, 1984; Hogan, 1985; Dhar and Das, 1990; 1991; 1994; Janssen, 1983).

In the present paper, we investigate the effect of randomness on the stability of deep water capillary gravity waves in the presence of air flowing over water, starting from a fourth order nonlinear evolution equation. Following Alber (1978), we first derive a spectral transport equation for narrowband Gaussian surface wave trains. This spectral transport equation is then used to study the stability of a uniform homogenous wave spectrum having a simple normal form, similar to small oblique long wavelength perturbations for a range of spectral widths. An expression for the growth rate of instability has been obtained which consists of two terms, one of which comes from the fourth order term in the evolution equation, which is responsible for wave induced mean flow and this produces a decrease in the growth rate. The growth rate of instability as a function of the effective modulation wave number $\left(\bar{k}\right)$ has been plotted for the different values of the bandwidth parameter $\left(\bar{\sigma}\right)$, air flow velocity ($v$), direction of perturbation (ϕ) surface tension (s) and the wave steepness $\left(\bar{a}_0\right)$.

The growth rate of instability is found to decrease with the increase of the spectral width $\left(\bar{\sigma}\right)$ and vanish when the spectral width increases beyond a certain critical value, which is not influenced by the fourth order terms in the evolution equation. An expression for the critical value of bandwidth has been obtained from the long wave length approximation of the dispersion relation whose order is found to be of the order of root mean square wave steepness $\bar{a}_0$. For vanishing spectral bandwidth and for perturbation direction $\phi = 0$, we find the deterministic maximum growth rate of instability which was obtained in Dhar and Das (1990), when the value of the non dimensional surface tension *s* is zero.

2. Derivation of the evolution equation

We take the common horizontal interface between air and water in the undisturbed state as the $z = 0$ plane, and air flows over water in the undisturbed state with a velocity *u* along the *x* axis. Let $z = \alpha\left(x, y, t\right)$ be the equation of the common interface at any time *t* in the perturbed state. We introduce the dimensionless quantities which are φ^*, φ'^* α^*, $\left(x^*, y^*, z^*\right)$, t^*, v^*. γ^* and s^* which are the perturbed velocity potential in water, perturbed velocity potential in air, surface elevation of the air water interface, space coordinates, time, air flow velocity, the ratio of densities ρ', ρ of air to water, and surface tension, respectively. These dimensionless quantities are related to the corresponding dimensional quantities by the following relations

$$\varphi^* = \sqrt{k_0^3 / g}\varphi, \qquad \varphi'^* = \sqrt{k_0^3 / g}\varphi', \qquad \alpha^* = k_0\alpha,$$

$$\left(x^*, y^*, z^*\right) = \left(k_0 x, k_0 y, k_0 z\right), \qquad t^* = \omega t, \tag{2.1}$$

$$v^* = \sqrt{k_0 / g}u, \qquad \gamma^* = \frac{\rho'}{\rho}, \qquad s^* = \frac{T k_0^2}{\rho g}$$

where k_0 is some characteristic wave number. In the future all the quantities will be written in their dimensionless form with their asterisks dropped.

The perturbed velocity potentials φ, φ' satisfy the following Laplace equations

$$\nabla^2\varphi = 0 \qquad \text{in} \qquad -\infty < z < \alpha, \tag{2.2}$$

$$\nabla^2\varphi' = 0 \qquad \text{in} \qquad \alpha < z < \infty. \tag{2.3}$$

The kinematic boundary conditions to be satisfied at the interface are

$$\frac{\partial\varphi}{\partial z} - \frac{\partial\alpha}{\partial t} = \frac{\partial\varphi}{\partial x}\frac{\partial\alpha}{\partial x} + \frac{\partial\varphi}{\partial y}\frac{\partial\alpha}{\partial y}, \qquad \text{when} \qquad z = \alpha, \tag{2.4}$$

$$\frac{\partial\varphi'}{\partial z} - \frac{\partial\alpha}{\partial t} - v\frac{\partial\alpha}{\partial x} = \frac{\partial\varphi'}{\partial x}\frac{\partial\alpha}{\partial x} + \frac{\partial\varphi'}{\partial y}\frac{\partial\alpha}{\partial y}, \qquad \text{when} \qquad z = \alpha. \tag{2.5}$$

The condition of continuity of pressure at the interface gives

$$\begin{aligned} &\frac{\partial\varphi}{\partial t} - \gamma\frac{\partial\varphi'}{\partial t} + (1-\gamma)\alpha - \gamma v\frac{\partial\varphi'}{\partial x} = -\frac{1}{2}\left[\left(\frac{\partial\varphi}{\partial x}\right)^2 + \left(\frac{\partial\varphi}{\partial y}\right)^2 + \left(\frac{\partial\varphi}{\partial z}\right)^2\right] + \\ &+\frac{\gamma}{2}\left[\left(\frac{\partial\varphi'}{\partial x}\right)^2 + \left(\frac{\partial\varphi'}{\partial y}\right)^2 + \left(\frac{\partial\varphi'}{\partial z}\right)^2\right] + s\left\{1 + \left(\frac{\partial\alpha}{\partial x}\right)^2 + \left(\frac{\partial\alpha}{\partial y}\right)^2\right\}^{\frac{-3}{2}} \times \\ &\times\left\{\left(\frac{\partial\alpha}{\partial x}\right)^2\frac{\partial^2\alpha}{\partial y^2} + \left(\frac{\partial\alpha}{\partial y}\right)^2\frac{\partial^2\alpha}{\partial x^2} - 2\frac{\partial\alpha}{\partial x}\frac{\partial\alpha}{\partial y}\frac{\partial^2\alpha}{\partial x\partial y} + \frac{\partial^2\alpha}{\partial x^2} + \frac{\partial^2\alpha}{\partial y^2}\right\}, \qquad \text{when} \qquad z = \alpha. \end{aligned} \tag{2.6}$$

Further, φ and φ' should satisfy the following conditions at infinity

$$\varphi \to 0 \qquad \text{as} \qquad z \to -\infty, \qquad \varphi' \to 0 \qquad \text{as} \qquad z \to \infty. \tag{2.7}$$

Since the disturbance is assumed to be a progressive wave we look for a solution to the above equations in the form

$$P = P_0 + \sum_{n=1}^{\infty}\left[P_n \exp in(kx - \omega t) + P_n^* \exp - in(kx - \omega t)\right] \tag{2.8}$$

where P stands for φ, φ', α and the star denotes complex conjugate. Here it is assumed that φ_0, φ'_0, φ_n, φ'_n, φ^*_n, φ'^*_n are functions of z, $x_1 = \varepsilon x$, $y_1 = \varepsilon y$, $t_1 = \varepsilon t$ and $\alpha_0, \alpha_n, \alpha^*_n$ are functions of x_1, y_1 and t_1, ε is the slowness parameter and the frequency ω and wave number k satisfy the following linear dispersion relation with $l = 0$

$$\lambda(\omega, k, l) = (1+\gamma)\omega^2 - 2\gamma\omega k v + \gamma k^2 v^2 - (1-\gamma)\left(k^2 + l^2\right)^{\frac{1}{2}} - s\left(k^2 + l^2\right)^{\frac{3}{2}} = 0. \tag{2.9}$$

We now suppose that the first harmonic linear wave, whose nonlinear evolution equation we are going to derive, has its wave number equal to k_0, the characteristic wave number. So we have $k = k_0 = 1$, and the linear dispersion relation (2.9) determining $\omega > 0$ becomes

$$(1+\gamma)\omega^2 - 2\gamma\omega\nu + \gamma\nu^2 - (1-\gamma) - s = 0\,. \tag{2.10}$$

From Eq.(2.10), we have $\omega_{\pm} = \left[\gamma v \pm \sqrt{1-\gamma^2-\gamma v^2+s(1+\gamma)}\right] \Big/ (1+\gamma)$ which corresponds to two modes and we designate them as positive and negative modes.

The linear stability ν should satisfy the following condition

$$|\nu| < \sqrt{\left[1-\gamma^2+s(1+\gamma)\right]\Big/\gamma}\,. \tag{2.11}$$

Thus our present analysis will remain valid as long as the dimensionless flow velocity of the air becomes less than the critical velocity $\sqrt{\left[1-\gamma^2+s(1+\gamma)\right]\Big/\gamma}$. For air flowing over water $\gamma = 0.00129$ and $s = 0.075$ so this critical value becomes 28.87.

By a standard procedure (Dhar and Das, 1990; Majumder and Dhar, 2009; 2009) we find that $\alpha = \alpha_{11} + \varepsilon\alpha_{12}$ satisfies the following fourth order nonlinear evolution equation

$$2i\frac{\partial\alpha}{\partial\tau} - \delta_1\frac{\partial^2\alpha}{\partial\xi^2} + \delta_2\frac{\partial^2\alpha}{\partial\eta^2} + i\delta_3\frac{\partial^3\alpha}{\partial\xi^3} + i\delta_4\frac{\partial^3\alpha}{\partial\xi\partial\eta^2} = \Psi\alpha \tag{2.12}$$

where $$\Psi\alpha = \wedge_1\alpha^2\alpha^* + i\wedge_2\alpha\alpha^*\frac{\partial\alpha}{\partial\xi} + i\wedge_3\alpha^2\frac{\partial\alpha^*}{\partial\xi} + \wedge_4\alpha H\frac{\partial\left(\alpha\alpha^*\right)}{\partial\xi}$$

where the coefficients $\delta_i, \wedge_i$ $(i = 1, 2, 3, 4)$ are given in the Appendix. Also ξ, η, τ denote the scaled variables $\xi = \varepsilon\left(x - c_g t\right), \eta = \varepsilon y, \tau = \varepsilon^2 t$, $c_g = \left(\frac{d\omega}{dk}\right)_{k=1}$ and H is Hilbert's transform operator given by

$$H(\overline{\psi}) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int\frac{(\xi'-\xi)\overline{\psi}(\xi',\eta')d\xi'd\eta'}{\left[(\xi'-\xi)^2+(\eta'-\eta)^2\right]^{3/2}}\,. \tag{2.13}$$

If we set $\gamma = v = 0$ and $s = 0$, then the evolution Eq.(2.12) reduces to an equation equivalent to Eq.(2.19) of Dysthe (1979).

3. Spectral transport equation

We assume that the evolution Eq.(2.12) for the complex amplitude $\alpha(\xi,\eta,\tau)$ describes the evolution of the wave train and $\overline{\xi}$ is a random function of ξ, η. Now we seek an equation for the slow variation of the two point space correlation function.

$$\rho\left(\overline{\xi_1},\overline{\xi_2},\tau\right) = \left\langle \alpha\left(\overline{\xi_1},\tau\right)\alpha^*\left(\overline{\xi_2},\tau\right)\right\rangle \tag{3.1}$$

where the angular bracket denotes an ensemble average and $\overline{\xi_1} = (\xi_1,\eta_1)$, $\overline{\xi_2} = (\xi_2,\eta_2)$. To obtain the equation for the two point correlation ρ from Eq.(2.12) we adopt the method followed by Alber (1978).

An equation is obtained from Eq.(2.12) at $\overline{\xi_1}$, by multiplying both sides of this by $\alpha^*\left(\overline{\xi_2},\tau\right)$. A second equation is obtained from the complex conjugate of Eq.(2.12) at $\overline{\xi_2}$, by multiplying both sides of this by $\alpha\left(\overline{\xi_1},\tau\right)$. Subtracting the second equation from the first and then taking the ensemble average, we get

$$\begin{aligned}
&2i\frac{\partial}{\partial\tau}\left\langle \alpha\left(\overline{\xi_1},\tau\right)\alpha^*\left(\overline{\xi_2},\tau\right)\right\rangle + \\
&-\delta_1\left[\frac{\partial^2}{\partial\xi_1^{\,2}}\left\langle \alpha\left(\overline{\xi_1},\tau\right)\alpha^*\left(\overline{\xi_2},\tau\right)\right\rangle - \frac{\partial^2}{\partial\xi_2^{\,2}}\left\langle \alpha\left(\overline{\xi_1},\tau\right)\alpha^*\left(\overline{\xi_2},\tau\right)\right\rangle\right] + \\
&+\delta_2\left[\frac{\partial^2}{\partial\eta_1^{\,2}}\left\langle \alpha\left(\overline{\xi_1},\tau\right)\alpha^*\left(\overline{\xi_2},\tau\right)\right\rangle + \frac{\partial^2}{\partial\eta_2^{\,2}}\left\langle \alpha\left(\overline{\xi_1},\tau\right)\alpha^*\left(\overline{\xi_2},\tau\right)\right\rangle\right] + \\
&+i\delta_3\left[\frac{\partial^3}{\partial\xi_1^{\,3}}\left\langle \alpha\left(\overline{\xi_1},\tau\right)\alpha^*\left(\overline{\xi_2},\tau\right)\right\rangle - \frac{\partial^3}{\partial\xi_2^{\,3}}\left\langle \alpha\left(\overline{\xi_1},\tau\right)\alpha^*\left(\overline{\xi_2},\tau\right)\right\rangle\right] + \\
&+i\delta_4\left[\frac{\partial^3}{\partial\xi_1\partial\eta_1^{\,2}}\left\langle \alpha\left(\overline{\xi_1},\tau\right)\alpha^*\left(\overline{\xi_2},\tau\right)\right\rangle + \frac{\partial^3}{\partial\xi_2\partial\eta_2^{\,2}}\left\langle \alpha\left(\overline{\xi_1},\tau\right)\alpha^*\left(\overline{\xi_2},\tau\right)\right\rangle\right] = \\
&= \left\langle \alpha\left(\overline{\xi_1},\tau\right)\alpha^*\left(\overline{\xi_2},\tau\right)\left[\Psi\left(\overline{\xi_1},\tau\right) - \Psi^*\left(\overline{\xi_2},\tau\right)\right]\right\rangle .
\end{aligned} \tag{3.2}$$

We now consider the following average coordinates $\overline{x}$ and spatial separation coordinates $\overline{r}$

$$\overline{x} = \tfrac{1}{2}\left(\overline{\xi}_1 + \overline{\xi}_2\right), \qquad \overline{r} = \overline{r} = \overline{\xi}_1 - \overline{\xi}_2 . \tag{3.3}$$

To evaluate fourth order correlation terms in Eq.(3.2) we assume that $\alpha\left(\overline{x},\tau\right)$ corresponds initially to a Gaussian random process and we further assume that the evolving random statistical amplitude field retains the same Gaussian statistical properties (Alber, 1978). For Gaussian statistics, the fourth order cumulant vanishes, allowing us to write the fourth order correlation in terms of product of second order correlations.

$$\begin{aligned}
&2i\frac{\partial\rho}{\partial\tau} - \delta_1\left(A_x^2 - B_x^2\right)\rho + \delta_2\left(A_y^2 - B_y^2\right)\rho + i\delta_3\left(A_x^3 - B_x^3\right)\rho + \\
&+i\delta_4\left(A_xA_y^2 - B_xB_y^2\right)\rho = 4\wedge_1\rho\sinh(M)\overline{a}^2\left(\overline{x},\tau\right) + \\
&+i\wedge_2\rho\cosh(M)\frac{\partial}{\partial x}\overline{a}^2\left(\overline{x},\tau\right) + i\wedge_2\frac{\partial\rho}{\partial x}\cosh(M)\overline{a}^2\left(\overline{x},\tau\right) + \\
&+2i\wedge_2\frac{\partial\rho}{\partial r_x}\sinh(M)\overline{a}^2\left(\overline{x},\tau\right) + 2i\wedge_3\rho\cosh(M)\frac{\partial}{\partial x}\overline{a}^2\left(\overline{x},\tau\right) + \\
&+\frac{\wedge_4}{\pi}\rho\sinh(M)\int\int_{-\infty}^{+\infty}\frac{(\xi'-x)\,d\xi'd\eta'}{\left[(\xi'-x)^2+(\eta'-y)^2\right]^{3/2}}\frac{\partial}{\partial\xi'}\overline{a}^2\left(\xi',\tau\right)
\end{aligned} \tag{3.4}$$

where A_x, B_x, A_y, B_y, M are defined by

$$A_x = \frac{1}{2}\frac{\partial}{\partial x} + \frac{\partial}{\partial r_x}, \qquad A_y = \frac{1}{2}\frac{\partial}{\partial y} + \frac{\partial}{\partial r_y}, \qquad B_x = \frac{1}{2}\frac{\partial}{\partial x} - \frac{\partial}{\partial r_x},$$
$$B_y = \frac{1}{2}\frac{\partial}{\partial y} + \frac{\partial}{\partial r_y}, \qquad M = \frac{1}{2} r_x \frac{\partial}{\partial x} + r_y \frac{\partial}{\partial y}, \tag{3.5}$$

and $\overline{a}^2(\overline{x},\tau)$ is the ensemble averaged mean-square amplitude given by

$$\overline{a}^2(\overline{x},\tau) = \left\langle \alpha(\overline{x},\tau)\alpha^*(\overline{x},\tau)\right\rangle. \tag{3.6}$$

The power spectral density $G(\overline{p},\overline{x},\tau)$ is defined as the Fourier transform of the two-point correlation function ρ

$$G(\overline{p},\overline{x},\tau) = \frac{1}{(2\pi)^2}\int\int_{-\infty}^{\infty} e^{-i\overline{p}\overline{r}}\rho\left(\overline{x}+\tfrac{1}{2}\overline{r},\overline{x}-\tfrac{1}{2}\overline{r},\tau\right)d\overline{r}. \tag{3.7}$$

Now the converse relation for ρ is given by

$$\rho = \int\int_{-\infty}^{\infty} G(\overline{p},\overline{x},\tau)e^{i\overline{p}\overline{r}}\, d\overline{p}, \tag{3.8}$$

and consequently $\overline{a}^2(\overline{x},\tau)$ is given by

$$\overline{a}^2(\overline{x},\tau) = \int\int_{-\infty}^{\infty} G(\overline{p},\overline{x},\tau)d\overline{p}. \tag{3.9}$$

Now taking the Fourier transform of Eq.(3.4) with respect to $\overline{r}$ we get the transport equation for $G(\overline{p},\overline{x},\tau)$.

$$\begin{aligned}
&2\frac{\partial G}{\partial \tau} - 2\delta_1 p_x \frac{\partial G}{\partial x} + 2\delta_2 p_y \frac{\partial G}{\partial y} + \delta_3\left(\tfrac{1}{4}\frac{\partial^3 G}{\partial x^3} - 3p_x^2\frac{\partial G}{\partial x}\right) + \\
&+\delta_4\left(\tfrac{1}{4}\frac{\partial^3 G}{\partial x \partial y^2} - p_y^2\frac{\partial G}{\partial x} - 2p_x p_y\frac{\partial G}{\partial y}\right) = 4\wedge_1 \sin\left(\overline{Q}\right)G\overline{a}^2 + \\
&+2\wedge_2\cos\left(\overline{Q}\right)G\frac{\partial \overline{a}^2}{\partial x} + \wedge_2\cos\left(\overline{Q}\right)\frac{\partial G}{\partial x}\overline{a}^2(\overline{x},\tau) + \\
&-2\wedge_2 p_x \sin\left(\overline{Q}\right)G\overline{a}^2 + 2\wedge_3\cos\left(\overline{Q}\right)G\frac{\partial \overline{a}^2}{\partial x} + \\
&+\frac{\wedge_4}{\pi}\sin\left(\overline{Q}\right)G\int\int_{-\infty}^{+\infty}\frac{(\xi'-x)\overline{a}^2_{\xi'}(\xi',\tau)d\xi' d\eta'}{\left[(\xi'-x)^2+(\eta'-y)^2\right]^{3/2}}
\end{aligned} \tag{3.10}$$

where $\bar{Q}$ is the operator given by

$$\bar{Q} = \frac{1}{2}\frac{\partial^2}{\partial x \partial p_x} + \frac{1}{2}\frac{\partial^2}{\partial y \partial p_y}. \tag{3.11}$$

Its space derivative operators $\frac{\partial}{\partial x}, \frac{\partial}{\partial y}$ operate only on $\bar{a}^2$, while the operators, $\frac{\partial}{\partial p_x}, \frac{\partial}{\partial p_y}$ operate on $\bar{G}$.

4. Stability analysis

The nonlinear spectral transport Eq.(3.10) has one basic solution

$$G = G_0(\bar{p}), \tag{4.1}$$

which is the random counterpart of uniform amplitude Stokes wave train of deterministic theory. $G_0(\bar{p})$ has Gaussian properties and is statistically uniform in space and time. To study the stability of this homogeneous solution to small amplitude oblique plane wave perturbation, we assume a perturbed solution to Eq.(3.10) in the form

$$G(\bar{p},\bar{x},\tau) = G_0(\bar{p}) + G_1(\bar{p},\bar{x},\tau), \tag{4.2}$$

$$\bar{a}^2 = \bar{a}_0^2 + \bar{a}_1^2(\bar{x},\tau). \tag{4.3}$$

From Eq.(3.9), we have the following relations

$$\bar{a}_0^2 = \int\int_{-\infty}^{\infty} G_0(\bar{p})d\bar{p}, \tag{4.4}$$

$$\bar{a}_1^2 = \int\int_{-\infty}^{\infty} G_1(\bar{p},\bar{x},\tau)d\bar{p}. \tag{4.5}$$

Substituting Eqs (4.2) and (4.3) in Eq.(3.10) and then linearizing we get

$$\begin{aligned}
&2\frac{\partial G_1}{\partial \tau} - 2\delta_1 p_x \frac{\partial G_1}{\partial x} + 2\delta_2 p_y \frac{\partial G_1}{\partial y} + \delta_3\left(\frac{1}{4}\frac{\partial^3 G_1}{\partial x^3} - 3p_x^2\frac{\partial G_1}{\partial x}\right) + \\
&+\delta_4\left(\frac{1}{4}\frac{\partial^3 G_1}{\partial x \partial y^2} - p_y^2\frac{\partial G_1}{\partial x} - 2p_x p_y\frac{\partial G_1}{\partial y}\right) = 4\wedge_1 \sin(\bar{Q})G_0\bar{a}_1^2 + \\
&+2\wedge_2\cos(\bar{Q})G_0\frac{\partial \bar{a}_1^2}{\partial x} + \wedge_2\cos(\bar{Q})\frac{\partial G_1}{\partial x}\bar{a}_0^2 + \\
&-2\wedge_2\sin(\bar{Q})p_x G_0\bar{a}_1^2 + 2\wedge_3\cos(\bar{Q})G_0\frac{\partial \bar{a}_1^2}{\partial x} + \\
&+\frac{\wedge_4}{\pi}\sin(\bar{Q})G_0\int\int_{-\infty}^{+\infty}\frac{(\xi'-x)\bar{a}_{1\xi'}^2(\bar{\xi}',\tau)}{\left[(\xi'-x)^2+(\eta'-y)^2\right]^{3/2}}d\xi' d\eta'.
\end{aligned} \tag{4.6}$$

Taking the Fourier transform of Eq.(4.6) with respect to $\overline{x}=(x,y)$ defined according to

$$\overline{G}_1(\tau) = \int\int_{-\infty}^{\infty} G_1(\overline{x},\tau) e^{i(lx+my)} d\overline{x}\,, \tag{4.7}$$

$$\overline{A}_1(\tau) = \int\int_{-\infty}^{\infty} \overline{a}_1^2(\overline{x},\tau) e^{i(lx+my)} d\overline{x}\,, \tag{4.8}$$

and then assuming τ - dependence of $\overline{G_1}(\tau)$, $\overline{A_1}(\tau)$ to be of the form $\exp(-i\Upsilon\tau)$, we get

$$\begin{aligned}
&\left[2\Upsilon + 2\delta_1 p_x l - 2\delta_2 p_y m + \delta_3\left(\tfrac{1}{4}l^3 + 3p_x^2 l\right) + \delta_4\left(\tfrac{1}{4}lm^2 + p_y^2 l + 2p_x p_y m\right)\right]\overline{G}_1 = \\
&= -4\wedge_1 \overline{A}_1 \sinh(L) G_0 - \wedge_2 l\overline{A}_1 \cosh(L) G_0 - \wedge_2 l\overline{a}_0^2 \overline{G}_1 + 2\wedge_2 \overline{A}_1 \sinh(L) p_x G_0 + \\
&-2\wedge_3 l\overline{A}_1 \cosh(L) G_0 + \frac{2\wedge_4 l^2}{\sqrt{l^2+m^2}} \overline{A}_1 \sinh(L) G_0
\end{aligned} \tag{4.9}$$

where the operator L is given by

$$L = \frac{l}{2}\frac{\partial}{\partial p_x} + \frac{m}{2}\frac{\partial}{\partial p_y}\,, \tag{4.10}$$

and the real part of Υ is the frequency shift, while the imaginary part of Υ is the growth rate of instability.

Similarly, taking the Fourier transform of Eq.(4.5) with respect to $\overline{x}$, we get

$$\overline{A}_1 = \int\int_{-\infty}^{\infty} \overline{G}_1 d\overline{p}\,. \tag{4.11}$$

Now in view of the relation (Alber, 1978)

$$\begin{aligned}
&\cosh\left(\frac{l}{2}\frac{\partial}{\partial p_x} + \frac{m}{2}\frac{\partial}{\partial p_y}\right) G_0(\overline{p}) = \frac{l}{2}\left[G_0\left(\overline{p} + \frac{l}{2}\overline{k}\right) + G_0\left(\overline{p} - \frac{l}{2}\overline{k}\right)\right], \\
&\sinh\left(\frac{l}{2}\frac{\partial}{\partial p_x} + \frac{m}{2}\frac{\partial}{\partial p_y}\right) G_0(\overline{p}) = \frac{l}{2}\left[G_0\left(\overline{p} + \frac{l}{2}\overline{k}\right) - G_0\left(\overline{p} - \frac{l}{2}\overline{k}\right)\right], \\
&\sinh\left(\frac{l}{2}\frac{\partial}{\partial p_x} + \frac{m}{2}\frac{\partial}{\partial p_y}\right) p_x G_0(\overline{p}) = \frac{l}{2} p_x \left[G_0\left(\overline{p} + \frac{l}{2}\overline{k}\right) + \right. \\
&\left. -G_0\left(\overline{p} - \frac{l}{2}\overline{k}\right)\right] + \frac{l}{4}\left[G_0\left(\overline{p} + \frac{l}{2}\overline{k}\right) + G_0\left(\overline{p} - \frac{l}{2}\overline{k}\right)\right]
\end{aligned} \tag{4.12}$$

where $\bar{k} = l\hat{x} + m\hat{y}$, and $\hat{x}$, $\hat{y}$ are unit vectors in the x and y directions, respectively, we get the following equation from (4.9)

$$\int\int_{-\infty}^{\infty} \bar{G}_1 d\bar{p} = \bar{A}_1 \int\int_{-\infty}^{\infty} \frac{g(\bar{p})G_0\left(\bar{p}+\frac{l}{2}\bar{k}\right)+h(\bar{p})G_0\left(\bar{p}-\frac{l}{2}\bar{k}\right)}{2\Upsilon+f(\bar{p})} d\bar{p} \tag{4.13}$$

where

$$\begin{aligned} f(\bar{p}) &= 2\delta_1 p_x l - 2\delta_2 p_y m + \delta_3\left(\tfrac{1}{4}l^3 + 3p_x^2 l\right) + \\ &+\delta_4\left(\tfrac{1}{4}lm^2 + p_y^2 l + 2p_x p_y m\right) + \wedge_2 l\bar{a}_0^2, \\ g(\bar{p}) &= -2\wedge_1 + \wedge_2 p_x - \wedge_3 l + \frac{\wedge_4 l^2}{\sqrt{l^2+m^2}}, \\ h(\bar{p}) &= 2\wedge_1 - \wedge_2 p_x - \wedge_3 l - \frac{\wedge_4 l^2}{\sqrt{l^2+m^2}}. \end{aligned} \tag{4.14}$$

using (4.11), Eq.(4.13) gives the following dispersion relation determining Υ

$$\int\int_{-\infty}^{\infty} \frac{g(\bar{p})G_0\left(\bar{p}+\frac{l}{2}\bar{k}\right)+h(\bar{p})G_0\left(\bar{p}-\frac{l}{2}\bar{k}\right)}{2\Upsilon+f(\bar{p})} d\bar{p} = 1. \tag{4.15}$$

Now let

$$\begin{aligned} f(\bar{p}) &= 2\left(\delta_1 p_x l - \delta_2 p_y m\right) + \varepsilon f_1(\bar{p}), \\ g(\bar{p}) &= -2\wedge_1 + \varepsilon g_1(\bar{p}), \\ h(\bar{p}) &= 2\wedge_1 + \varepsilon h_1(\bar{p}) \end{aligned} \tag{4.16}$$

where order ε terms are fourth order contributions and where $f_1(\bar{p})$, $g_1(\bar{p})$, $h_1(\bar{p})$ are given by

$$\begin{aligned} f_1(\bar{p}) &= \delta_3\left(\tfrac{1}{4}l^3 + 3p_x^2 l\right) + \delta_4\left(\tfrac{1}{4}lm^2 + p_y^2 l + 2p_x p_y m\right) + \wedge_2 l\bar{a}_0^2, \\ g_1(\bar{p}) &= \wedge_2 p_x - \wedge_3 l + \frac{\wedge_4 l^2}{\sqrt{l^2+m^2}}, \\ h_1(\bar{p}) &= -\wedge_2 p_x - \wedge_3 l - \frac{\wedge_4 l^2}{\sqrt{l^2+m^2}}. \end{aligned} \tag{4.17}$$

Substituting Eqs (4.16) in Eq.(4.15) and then keeping only $0(\varepsilon)$ terms, the equation determining Υ can be expressed as

$$\wedge_1 \int\int_{-\infty}^{\infty} \frac{G_0\left(\bar{p}+\frac{l}{2}\bar{k}\right)-G_0\left(\bar{p}-\frac{l}{2}\bar{k}\right)}{\Upsilon+\left(\delta_1 l p_x-\delta_2 m p_y\right)} dp_x dp_y + 1 =$$

$$= \frac{\varepsilon\wedge_1}{2}\int\int_{-\infty}^{\infty} \frac{\left[G_0\left(\bar{p}+\frac{l}{2}\bar{k}\right)-G_0\left(\bar{p}-\frac{l}{2}\bar{k}\right)\right] f_1(\bar{p})}{\left[\Upsilon+\left(\delta_1 l p_x-\delta_2 m p_y\right)\right]^2} dp_x dp_y + \tag{4.18}$$

$$+\frac{\varepsilon}{2}\int\int_{-\infty}^{\infty} \frac{g_1(\bar{p})G_0\left(\bar{p}+\frac{l}{2}\bar{k}\right)+h_1(\bar{p})G_0\left(\bar{p}-\frac{l}{2}\bar{k}\right)}{\Upsilon+\left(\delta_1 l p_x-\delta_2 m p_y\right)} dp_x dp_y .$$

Here for $G_0(\bar{p})$ we shall assume a simple spectrum which facilitates the solution of Eq.(4.18), in the two dimensional normal spectrum

$$G_0(\bar{p}) = \frac{\bar{a}_0^2}{2\pi\sigma^2}\exp\left[-\frac{p_x^2+p_y^2}{2\sigma^2}\right]. \tag{4.19}$$

4.1. Roots of the dispersion relation considering third order terms only

We first determine the roots of the dispersion relation (4.18) neglecting order ε terms, which originate from the fourth order terms in the evolution Eq.(2.12). Equation (4.18) then becomes the following by the use of expression (4.19) for $G_0(\bar{p})$

$$\frac{\bar{a}_0^2\wedge_1}{\sqrt{2\pi}\sigma|k|\delta_1}\left[\int_{-\infty}^{\infty}\frac{\exp\left(-t^2\right)}{\frac{\Upsilon-\omega_0}{\sqrt{\sigma}|k|\delta_1}-t}dt-\int_{-\infty}^{\infty}\frac{\exp\left(-t^2\right)}{\frac{\Upsilon+\omega_0}{\sqrt{\sigma}|k|\delta_1}-t}dt\right]+1=0 \tag{4.20}$$

where

$$\omega_0 = \tfrac{1}{2}\left(\delta_1 l^2-\delta_2 m^2\right),$$

$$. \tag{4.21}$$

$$k = l\hat{x} - m\left(\frac{\delta_2}{\delta_1}\right)\hat{y} = |k|\left(\hat{x}\cos\theta+\hat{y}\sin\theta\right).$$

Now setting $\Upsilon=\Upsilon_r+i\Upsilon_i$ in Eq.(4.1.1), where Υ_r and Υ_i are the real and imaginary parts of Υ, and then separating into real and imaginary parts we get two equations for the determination of the two quantities Υ_r and Υ_i. It is found that one of the two equations is satisfied identically with $\Upsilon=i\Upsilon_i$. Hence we can suppose that the real part of Υ determined by Eq.(4.1.1) is zero. So setting $\Upsilon=i\Upsilon_i$, where Υ_i is real, Eq.(4.1.1) can be expressed as

$$1+\frac{i\overline{a}_0^2\sqrt{\pi}\wedge_1}{\sqrt{2}\sigma|k|\delta_1}\left[\omega(z)-\omega\left(-z^*\right)\right]=0 \tag{4.22}$$

where

$$\omega(z)=\frac{i}{\pi}\int_{-\infty}^{\infty}\frac{e^{-t^2}}{z-t}dt, \tag{4.23}$$

$$z=\frac{\omega_0}{\sqrt{2}\sigma|k|\delta_1}+\frac{i\Upsilon_i}{\sqrt{2}\sigma|k|\delta_1}, \tag{4.24}$$

and z^* is the complex conjugate of z.

4.2. Roots of the dispersion relation including higher order terms

The dispersion relation (4.18) including order ε terms can be expressed as

$$\chi(\Upsilon)=\varepsilon\Phi(\Upsilon) \tag{4.25}$$

where

$$\chi(\Upsilon)=1+\frac{i\overline{a}_0^2\sqrt{\pi}\wedge_1}{\sqrt{2}\sigma|k|\delta_1}\left[\omega(x)-\omega(-x')\right] \tag{4.26}$$

and

$$\Phi(\Upsilon)=\tfrac{1}{2}\wedge_1 I_1(x')-\tfrac{1}{2}\wedge_1 I_2(x)+\tfrac{1}{2}I_3(x')+\tfrac{1}{2}I_4(x), \tag{4.27}$$

x, x' I_1, I_2, I_3, I_4 being given by

$$x=\frac{\Upsilon+\omega_0}{\sqrt{2}\sigma|k|\delta_1},\qquad x'=\frac{\Upsilon-\omega_0}{\sqrt{2}\sigma|k|\delta_1},$$

$$I_1=\frac{\overline{a}_0^2}{\sqrt{\pi}\left(\sqrt{2}\sigma|k|\delta_1\right)^2}\int_{-\infty}^{\infty}\left[\left(a_1+a_2\sigma^2\right)-\sqrt{2}a_3\sigma t+2a_4\sigma^2t^2\right]\frac{\exp\left(-t^2\right)}{\left(x'-t\right)^2}dt,$$

$$I_2=\frac{\overline{a}_0^2}{\sqrt{\pi}\left(\sqrt{2}\sigma|k|\delta_1\right)^2}\int_{-\infty}^{\infty}\left[\left(a_1+a_2\sigma^2\right)+\sqrt{2}a_3\sigma t+2a_4\sigma^2t^2\right]\frac{\exp\left(-t^2\right)}{\left(x-t\right)^2}dt, \tag{4.28}$$

$$I_3=\frac{\overline{a}_0^2}{\sqrt{2\pi}\sigma|k|\delta_1}\int_{-\infty}^{\infty}\left(b_1-\sqrt{2}b_2\sigma t\right)\frac{\exp\left(-t^2\right)}{\left(x'-t\right)}dt,$$

$$I_4=\frac{\overline{a}_0^2}{\sqrt{2\pi}\sigma|k|\delta_1}\int_{-\infty}^{\infty}\left(c_1+\sqrt{2}b_2\sigma t\right)\frac{\exp\left(-t^2\right)}{\left(x-t\right)}dt.$$

The constants a_1, a_2, a_3, a_4, b_1, b_2, c_1 appearing in Eqs (4.28) are given in the Appendix.

As the root of Eq.(4.25) with order ε terms set equal to zero is $i\Upsilon_i$, where Υ_i is real and is given by Eq.(4.22), we take

$$\Upsilon = i\Upsilon_i + \varepsilon\Upsilon_1, \tag{4.29}$$

as the root of Eq.(4.25). Substituting Eq.(4.29) in Eq.(4.25) we find that Υ_1 in the lowest order is given by

$$\Upsilon_1 = \frac{\Phi(i\Upsilon_i)}{\chi'(i\Upsilon_i)}. \tag{4.30}$$

Evaluating $\Phi(i\Upsilon_i)$ and $\chi'(i\Upsilon_i)$ from the expressions (4.26) and (4.27) for $\Phi(\Upsilon)$ and $\chi(\Upsilon)$, respectively, we get

$$\begin{aligned} \Phi(i\Upsilon_i) &= \tfrac{1}{2}\wedge_1(k_1 - k_2) + \tfrac{1}{2}(k_3 + k_4), \\ \chi'(i\Upsilon_i) &= \frac{i\bar{a}_0^2\sqrt{\pi}\wedge_1}{\left(\sqrt{2}\sigma|k|\delta_1\right)^2}\left[\omega'(z) - \omega'(-z^*)\right]. \end{aligned} \tag{4.31}$$

The expressions for k_1, k_2, k_3, k_4 are given in the Appendix.

From the expression (4.30) for Υ_1 it is found that the imaginary part of Υ given by Eq.(4.29) is the following, which is therefore the growth rate of instability Γ, is given by

$$\begin{aligned} \Gamma &= \mathrm{Im}(\Upsilon) = \Upsilon_i + \mathrm{Im}(\Upsilon_1) \\ &= 2\sqrt{2}\delta_1\bar{\sigma}\bar{K}\bar{a}_0^2\left[\beta' - \frac{\wedge_4 k\bar{a}_0}{\sqrt{2}\wedge_1}\mathrm{Im}\{\omega(z)\}\left[\alpha'\mathrm{Re}\{\omega(z)\} - \beta'\mathrm{Im}\{\omega(z)\}\right]^{-1}\times\right. \\ &\left.\times\cos^2\phi\left(\cos^2\phi - \frac{\delta_2}{\delta_1}\sin^2\phi\right)^{-\frac{1}{2}}\right] \end{aligned} \tag{4.32}$$

where

$$\bar{K} = \sqrt{\frac{l^2 + m^2}{2\bar{a}_0^2}}\left(\cos^2\phi - \frac{\delta_2}{\delta_1}\sin^2\phi\right)^{\frac{1}{2}},$$

$$\bar{\sigma} = \frac{\sigma}{\sqrt{2\bar{a}_0^2}}\left[\frac{\cos^2\phi + \left(\frac{\delta_2}{\delta_1}\right)^2\sin^2\phi}{\cos^2\phi - \left(\frac{\delta_2}{\delta_1}\right)^2\sin^2\phi}\right]^{\frac{1}{2}}, \tag{4.33}$$

$$z = \alpha' + i\beta', \qquad \frac{m}{l} = \tan\phi.$$

The term involving $\wedge_4$ in the expression for Γ is the only higher order contribution, i.e., the fourth order contribution, and this originates from the term involving $\wedge_4$ in the evolution Eq.(2.12), which is known to be produced by the wave induced mean flow.

$\overline{K}$ and $\overline{\sigma}$ appearing in the expression (4.32) for Γ can be considered as the effective modulation wave number and effective bandwidth parameter respectively. Here the effective modulation wave number has been defined as a modulation wave number multiplied by a factor. A similar definition has been given for the bandwidth parameter. These two effective parameters have been introduced instead of the original parameter only to get a rather simplified expression for the growth rate of instability as given by Eq.(4.32). The growth of instability Γ as a function of the effective modulation wave number $\overline{K}$ has been plotted in Figs 1-4 for different values of the effective bandwidth parameter $\overline{\sigma}$, wind velocity v and two different values of the perturbation angle ϕ, mean square wave steepness $\overline{a}_0^2$ and for the dimensionless surface tension $s = 0.075$. In each of the figures from 1-4 both third order and fourth order results have been shown. In each of the figures curves labeled 1-8, $\overline{\sigma}$ and v have values as shown in the following table.

Curve number	1	2	3	4	5	6	7	8
$\overline{\sigma}$	*1*	*1*	*0.75*	*0.75*	*0.5*	*0.5*	*0*	*0*
v	*12*	*0*	*12*	*0*	*12*	*0*	*12*	*0*

where the values of the density ratio γ of air to water are *0* and *0.00129* corresponding to $v = 0$ and *12*, respectively.

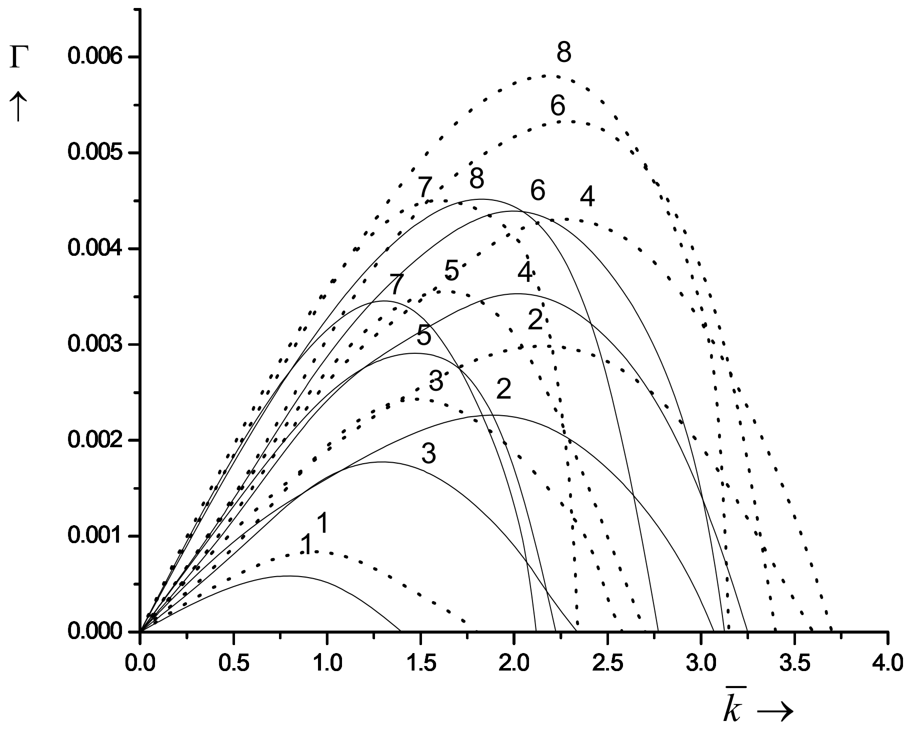

Fig.1. The growth rate of instability Γ as a function of the effective modulation wave number $\overline{K}$, $\overline{a}_0 = 0.1$, $\phi = 0^0$; ……….. : third order result; ____________ : fourth order result.

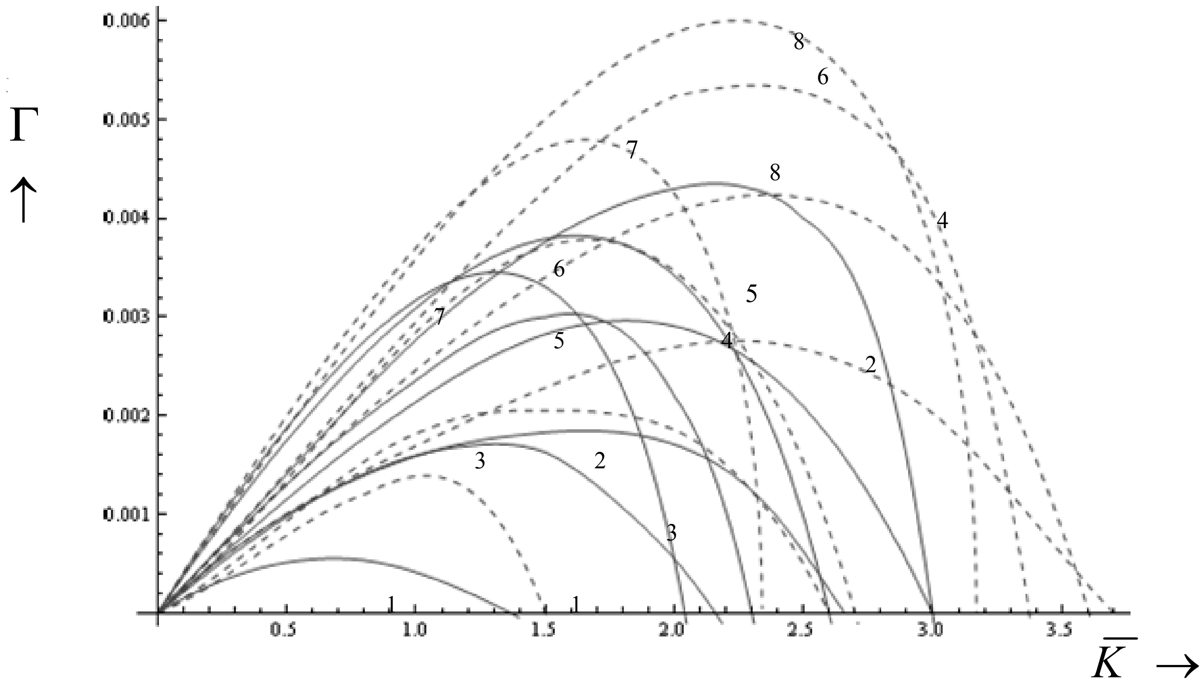

Fig.2. The growth rate of instability Γ as a function of the effective modulation wave number $\overline{K}$, $\overline{a}_0 = 0.1$, $\phi = 30^0$; ……….. : third order result; ____________ : fourth order result.

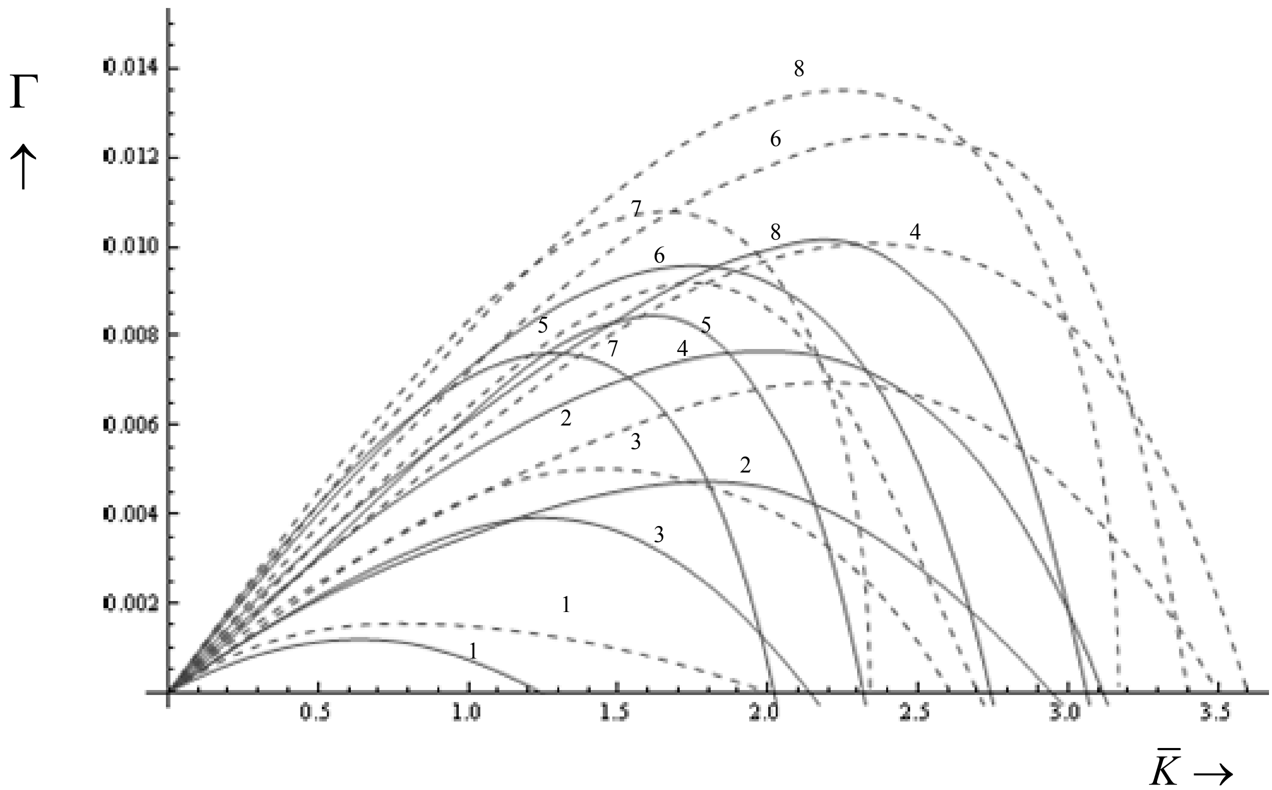

Fig.3. The growth rate of instability Γ as a function of the effective modulation wave number $\overline{K}$, $\overline{a}_0 = 0.15$, $\phi = 0^0$; ……….. : third order result; ____________ : fourth order result.

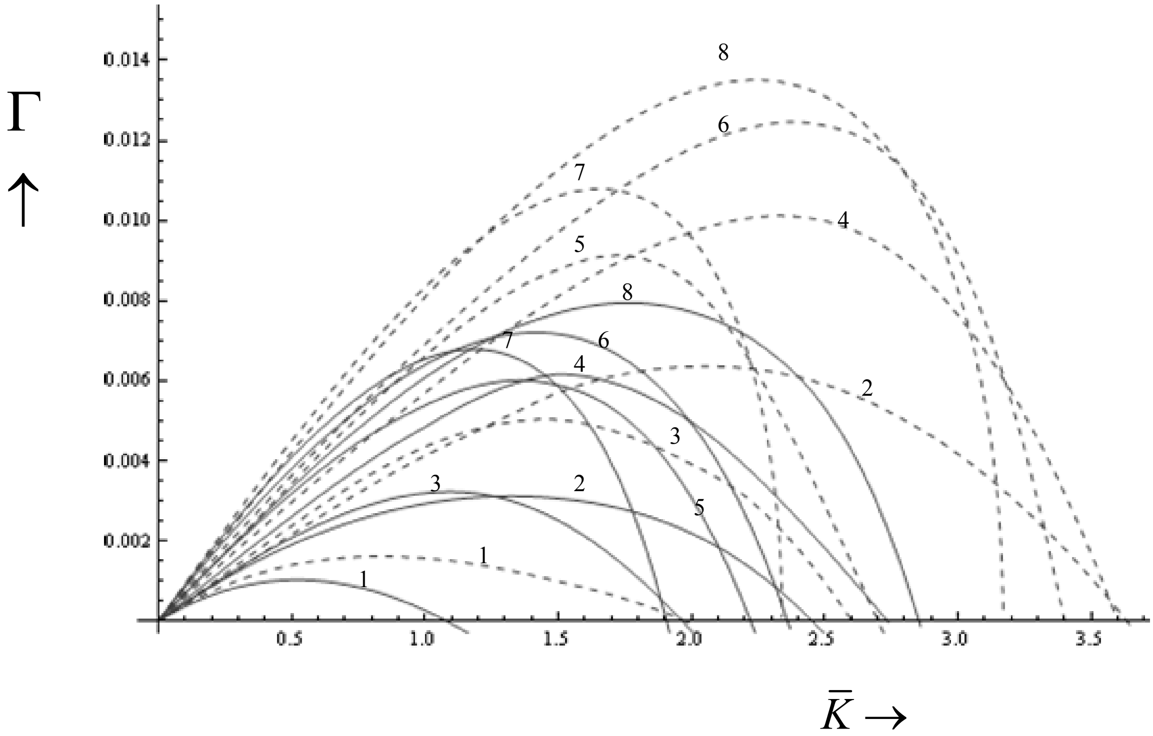

Fig.4. The growth rate of instability Γ as a function of the effective modulation wave number $\overline{K}$, $\bar{a}_0 = 0.15$, $\phi = 30^0$; ………. : third order result; ___________ : fourth order result.

From these figures it is seen that the growth rate of instability decreases with the increase of the spectral bandwidth, and the higher order, i.e., fourth order term produces a further decrease. Now the instability vanishes if the spectral width increases beyond a certain critical value. To find this critical value of the bandwidth we proceed as follows.

As is evident from figures the critical slope of amplification curve diminishes with the increase in the spectral bandwidth. From this we find that a condition of stability is $\left(\frac{\partial \Gamma}{\partial \lambda}\right)_{\lambda=0} < 0$ where $\Gamma = \mathrm{Im}(\Upsilon)$ and $\lambda = \sqrt{l^2 + m^2}$ is the modulation wave number. Now $\left(\frac{\partial \Gamma}{\partial \lambda}\right)_{\lambda=0}$ can be obtained from the long wave length approximation of the dispersion relation (4.25). By exact calculation it is found that the contribution of the fourth order terms in $\left(\frac{\partial \Gamma}{\partial \lambda}\right)_{\lambda=0}$ is zero. After calculation from the third order term we find the following expression for $\left(\frac{\partial \Gamma}{\partial \lambda}\right)_{\lambda=0}$

$$\left(\frac{\partial \Gamma}{\partial \lambda}\right)_{\lambda=0} = \left(\frac{\partial \Upsilon_i}{\partial \lambda}\right)_{\lambda=0} = \frac{\sqrt{2}\sigma\delta_1}{\sqrt{\pi}} \left\{\cos^2\phi + \left(\delta_2 / \delta_1\right)^2 \sin^2\phi\right\}^{1/2} \left(1 - \frac{2\sigma^2 |k|^2 \delta_1^2}{4 \wedge_1 \bar{a}_0^2 \omega_0}\right). \tag{4.34}$$

Therefore the condition of stability $\left(\frac{\partial \Gamma}{\partial \lambda}\right)_{\lambda=0} < 0$ gives

$$\sigma > \frac{\sqrt{2\wedge_1}}{\delta_1}\bar{a}_0 \left[\frac{\delta_1 l^2 - \delta_2 m^2}{2\left\{l^2 + \left(\frac{\delta_2}{\delta_1}\right)^2 m^2\right\}}\right]^{1/2}, \tag{4.35}$$

which states that there the stability of the bandwidth exceeds a certain critical value σ_c given by

$$\sigma_c = \frac{\sqrt{2\wedge_1}}{\delta_1}\bar{a}_0 \left[\frac{\delta_1 l^2 - \delta_2 m^2}{2\left\{l^2 + \left(\frac{\delta_2}{\delta_1}\right)^2 m^2\right\}}\right]^{\frac{1}{2}}. \tag{4.36}$$

This critical value is of the order of $\bar{a}_0$.

5. The limit of vanishing bandwidth

The deterministic growth rate of instability can be obtained from Eq.(4.32) by making $\sigma \to 0$. From Eq.(4.25) we find that $\sigma \to 0$ implies $z \to \infty$. So we can use the asymptotic expansion of $\omega(z)$ for $z \to \infty$, which is the following

$$\omega(z) = \frac{i}{\sqrt{\pi}z}\left(1 + \frac{1}{2z^2} + \frac{3}{4z^4} + \ldots\right). \tag{5.1}$$

For the vanishing bandwidth the expression Γ is given by

$$\Gamma = \Upsilon_{i0} - \frac{\sqrt{2}\wedge_4 \delta_1 \bar{K}^3 \bar{a}_0^5 \cos^2\phi}{\Upsilon_{i0}\left\{\cos^2\phi - \left(\frac{\delta_2}{\delta_1}\right)\sin^2\phi\right\}^{\frac{1}{2}}} \tag{5.2}$$

where

$$\Upsilon_{i0} = \sqrt{2}\delta_1 \bar{K}\bar{a}_0^2 \left(\frac{\wedge_1}{\delta_1} - \frac{\bar{k}^2}{2}\right)^{1/2}, \tag{5.3}$$

which is of the third order, i.e., lower order growth rate of instability for the vanishing bandwidth. This can be made identical with the deterministic growth rate, if $2\bar{a}_0^2$ is replaced by $\bar{a}_0^2$, the mean-square wave steepness (Alber, 1978).

For $\phi = 0$, the expression for Γ given by Eq.(5.2) in which we replace $2\bar{a}_0^2$ by $\bar{a}_0^2$ becomes

$$\Gamma = \frac{1}{2}\left[\frac{\delta_1 l^2}{2}\left(\frac{\wedge_1}{4}\bar{a}_0^2 - \frac{\delta_1}{2}l^2 - \frac{\wedge_4 l}{4}\bar{a}_0^2\right)\right]^{1/2}. \tag{5.4}$$

From Eq.(5.4) we get the following expression for the maximum growth rate Γ_M of instability

$$\Gamma_M = \frac{\wedge_1 \bar{a}_0^2}{8}\left[1 - \frac{\wedge_4 \bar{a}_0}{\sqrt{4\wedge_1\delta_1}}\right]. \tag{5.5}$$

If we set $\gamma = \nu = 0$ and $s = 0$, Eq.(5.5) reduces to Eq.(3.10) of Dysthe (1979).

6. Conclusion

Dysthe (1979) has shown that a fourth order nonlinear evolution equation, which is one order higher than the lowest order evolution equation, is a good starting point for studying the nonlinear surface waves in deep water for waves of wave steepness up to *0.25*.

In the present paper we investigate the effect of randomness on the stability of deep water capillary gravity waves in the presence of air flowing over water, starting from a fourth order nonlinear evolution equation. Following Alber (1978), we first derive a spectral transport equation for narrow band Gaussian surface wave trains. This spectral transport equation is then used to study the stability of a uniform homogenous wave spectrum having a simple normal form, similar to small oblique long wavelength perturbations for a range of spectral widths. An expression for the growth rate of instability has been obtained which consists of two terms, one of which comes from the fourth order term in the evolution equation, which is responsible for the wave induced mean flow. This higher order contribution in the expression for the growth rate of instability produces a decrease of the growth rate. The growth rate of instability is found to decrease with the increase in the spectral width, and ultimately the instability vanishes if the spectral width exceeds a certain critical value. This critical value remains unaffected by the fourth order terms in the evolution equation, and an expression for this critical value is obtained from the long wave length approximation of the dispersion relation. Since the evolution equation from which we have started our analysis is valid for narrow spectral band, stability beyond the critical spectral width does not imply stability for large spectral band widths. For vanishing spectral band widths we recover the deterministic growth rate of instability obtained in my paper (Dhar and Das, 1990). From this growth rate of instability the maximum growth rate of instability has been obtained. This again reduces to the expression of the maximum growth rate of instability obtained by Dysthe (1979) for $\gamma = \nu = 0$, and $s = 0$. Dysthe found that this maximum growth rate of instability is in good agreement with the exact result of Longuet – Higgins (1978a; 1978b). Here we can conclude that our results in the limit of vanishing bandwidth agreed favorably with the result of Longuet – Higgins (1978a; 1978b).

Appendix-1

The coefficients of Eq.(2.12)

$$\delta_1 = -\frac{dc_g}{dk}, \qquad \delta_2 = \frac{1-\gamma+3s}{\lambda_\omega}, \qquad \delta_3 = \frac{2i}{\lambda_\omega}\left[(1+\gamma)c_g\frac{dc_g}{dk} - s\right],$$

$$\delta_4 = \frac{2i}{\lambda_\omega}\left[(1+\gamma)c_g\frac{(1-\gamma+3s)}{\lambda_\omega} + \frac{1-\gamma-3s}{2}\right],$$

$$\wedge_1 = \frac{\beta_1}{\lambda_\omega}, \qquad \wedge_2 = \frac{2}{\lambda_\omega}\left[\beta_3 - \beta_2 c_g + 2(1+\gamma)c_g\,\beta_1/\lambda_\omega - 6s\right],$$

$$\wedge_3 = \frac{2}{\lambda_\omega}\left[\beta_4 + (1+\gamma)c_g\beta_1/\lambda_\omega\right], \qquad \wedge_4 = \frac{8}{\lambda_\omega}\left[\omega^2 + \gamma(\omega-v)^2\right]$$

where

$$\beta_1 = 4\left[\omega^2 + \gamma(\omega-v)^2 + \frac{\left\{\omega^2 - \gamma(\omega-v)^2\right\}^2}{1-\gamma-2s} + \tfrac{3}{4}s\right],$$

$$\beta_2 = 8\left[\frac{\omega+\gamma(\omega-v)}{2} + \frac{\left\{\omega-\gamma(\omega-v)\right\}\left\{\omega^2-\gamma(\omega-v)^2\right\}}{1-\gamma-2s} - \frac{\left\{\omega+\gamma(\omega-v)\right\}\left\{\omega^2-\gamma(\omega-v)^2\right\}^2}{(1-\gamma-2s)^2}\right],$$

$$\beta_3 = -2\left[\omega^2 + \gamma(\omega-v)^2 + \frac{\left\{4\gamma v(\omega-v) + (1-\gamma) + 12s\right\}\left\{\omega^2-\gamma(\omega-v)^2\right\}^2}{(1-\gamma-2s)^2} + \right.$$
$$+2\left\{\omega^2 + \gamma(\omega-v)(\omega-2v)\right\} + \frac{\left\{\omega^2-\gamma(\omega-v)^2\right\}\left\{\omega^2-\gamma(\omega-v)(\omega-3v)\right\}}{(1-\gamma-2s)} +$$
$$\left. +2\frac{\left\{\omega^2-\gamma(\omega-v)(\omega-2v)\right\}\left\{\omega^2-\gamma(\omega-v)^2\right\}}{(1-\gamma-2s)}\right],$$

$$\beta_4 = -2\left[\omega^2 + \gamma(\omega-v)^2 + \frac{\left\{\omega^2-\gamma(\omega-v)^2\right\}^2}{1-\gamma-2s}\right],$$

$$\lambda_\omega = 2\left[(1+\gamma)\omega - \gamma v\right].$$

Appendix-2

The coefficients $a_1,\ a_2,\ a_3,\ a_4,\ b_1,\ b_2,\ c_1$ appearing in the expression (4.28)

$$a_1 = \delta_3 l^3 + \delta_4 l m^2 + \wedge_2 l \bar{a}_0^2,$$

$$a_2 = 3\delta_3 l \sin^2\theta - 2\delta_4 m \sin\theta\cos\theta + \delta_4\cos^2\theta\,,$$

$$a_3 = 2\delta_4 lm\sin\theta - \left(3\delta_3 l^2 + \delta_4 m^2\right)\cos\theta,$$

$$a_4 = 3\delta_3 l\cos^2\theta + 2\delta_4 m\sin\theta\cos\theta + \delta_4 l\sin^2\theta,$$

$$b_1 = -\tfrac{1}{2}\wedge_2 l - \wedge_3 l + \frac{\wedge_4 l^2}{\sqrt{l^2+m^2}},$$

$$b_2 = \wedge_2\cos\theta\,,$$

$$c_1 = -\tfrac{1}{2}\wedge_2 l - \wedge_3 l - \frac{\wedge_4 l^2}{\sqrt{l^2+m^2}},$$

where θ is defined by the second equation of (4.21).

Appendix-3

Expressions for k_1, k_2, k_3, k_4 appearing in (4.31)

$$k_1 = \frac{a_4\bar{a}_0^2}{\left(\left|\hat{k}\right|\delta_1\right)^2} - \frac{i\sqrt{\pi}\bar{a}_0^2}{\left(\sqrt{2}\sigma\delta_1\left|\hat{k}\right|\right)^2}\left(\sqrt{2}a_3\sigma + 4a_4\sigma^2 z^*\right)\omega\left(-z^*\right) +$$
$$+\frac{i\sqrt{\pi}\bar{a}_0^2}{\left(\sqrt{2}\sigma\delta_1\left|\hat{k}\right|\right)^2}\left(a_1 + a_2\sigma^2 + \sqrt{2}a_3\sigma z^* + 2a_4\sigma^2 z^{*2}\right)\omega'\left(-z^*\right),$$

$$k_2 = \frac{a_4\bar{a}_0^2}{\left(\left|\hat{k}\right|\delta_1\right)^2} + \frac{i\sqrt{\pi}\bar{a}_0^2}{\left(\sqrt{2}\sigma\delta_1\left|\hat{k}\right|\right)^2}\left(\sqrt{2}a_3\sigma + 4a_4\sigma^2 z^*\right)\omega\left(z\right) +$$
$$+\frac{i\sqrt{\pi}\bar{a}_0^2}{\left(\sqrt{2}\sigma\delta_1\left|\hat{k}\right|\right)^2}\left(a_1 + a_2\sigma^2 + \sqrt{2}a_3\sigma z^* + 2a_4\sigma^2 z^{*2}\right)\omega'\left(z\right),$$

$$k_3 = \frac{b_2\bar{a}_0^2}{\left|\hat{k}\right|\delta_1} - \frac{i\sqrt{\pi}\bar{a}_0^2}{\sqrt{2}\sigma\delta_1\left|\hat{k}\right|}\left(b_1 + \sqrt{2}b_2\sigma z^*\right)\omega\left(-z^*\right),$$

$$k_4 = -\frac{b_2\bar{a}_0^2}{\left|\hat{k}\right|\delta_1} - \frac{i\sqrt{\pi}\bar{a}_0^2}{\sqrt{2}\sigma\delta_1\left|\hat{k}\right|}\left(c_1 + \sqrt{2}b_2\sigma z\right)\omega\left(z\right).$$

Nomenclature

g – acceleration due to gravity
H – Hilbert's transform operator
k – wave number
P – general solution to Eqs (2.4) - (2.6)
s – dimensionless surface tension
t – time
v – air flow velocity
(x, y, z) – space coordinates
α – elevation of the air water interface
$\Delta\omega$ – frequency shift
$\delta_i\,(i = 1,2,3,4)$, $\beta_i\,(i = 1,2,3,4)$, $\wedge_i\,(i = 1,2,3,4)$ – coefficients given in the Appendix
ε – slowness parameter
γ – ratio of densities of air to water
η_0 – wave steepness
(η', θ') – small real perturbations of amplitude and phase
ξ, η, τ – transformed variables
ω – frequency
Υ – perturbed frequency at marginal stability

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