Repetitive process based design of PD-type iterative learning control laws for batch processes with time-delays

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Abstract

This paper uses linear matrix inequality (LMI) techniques to design an iterative learning control (ILC) law or a class of batch processes with state delays. The design procedure uses the stability theory for repetitive processes, where the design problem is first converted into one for a parallel interconnection of repetitive processes with delays. This setting allows the computation of the PD-type control law gain matrices, and the Kalman-Yakubovich-Popov (KYP) lemma ensures batch-to-batch error convergence over the complete frequency domain. A numerical example illustrates the paper developments and proves that the design is effective and brings some advantages when compared to the existing alternatives.

1 Introduction

Many industries, e.g., the fine chemical industry, machinery manufacturing, electronic technology, pharmaceutical engineering, and agricultural products processing, use batch processing control systems to obtain high-quality products with high manufacturing efficiency. Therefore, the development of advanced batch process control technologies and performance assessment methods, see, e.g.,in [3, 7], can result in improved productivity and quality of manufacturing-related industries.

   Recent developments in batch process control technologies include using output information for each batch to improve the control for the subsequent one. Given that the processing time for each batch is finite and fixed, this is an application area for ILC. See, e.g., [1, 7] for early research results in this application area. As in other ILC applications, a variable is needed to specify or index a batch, and another to describe the dynamics within a batch. In this paper, the batch

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index is denoted by the nonnegative integer k, and t describes the finite duration dynamics within a batch. Moreover, the time taken to process a batch is denoted by $\alpha < \infty$.

In batch processing, there are two directions of information propagation, i.e., within each batch, where the duration is finite, and from batch to batch. The dynamics are defined over $(k, t) = [0\infty] \times [0, \alpha]$. Hence ILC batch processing is a member of the class of 2D systems. In particular, the dynamics considered can be written as a repetitive process, i.e., a distinct class of 2D systems [5]. This setting allows simultaneous design for batch-to-batch (k) and batch dynamics (t).

The literature demonstrates that a P-type ILC law (where the control signal is proportional to the previous batch error only) can effectively control the average error of the tracking reference. Still, the local maximum error control effect is not ideal, e.g., in many industrial processes, the maximum local error may even result in wasted production. Therefore, it is necessary to improve the control law to reduce the maximum error, where differential (D)-type control can improve the dynamic characteristics of the control system. e.g., reducing overshoot, shortening regulation time, increasing proportional control, and enhancing control accuracy.

This paper focuses on the design of PD-type ILC laws. The repetitive process setting is used for analysis control, and the novel contributions are as follows:

- the design of a PD-type ILC law in the frequency domain for a class of linear differential systems with time delays;

- batch-to-batch error convergence conditions as LMIs.

Throughout this paper, $\mathbb{R}^n$ denotes n-dimensional Euclidean space, $\mathbb{R}^{n \times m}$ is the set of all $n \times m$ real matrices. For symmetric matrices, $X \prec Y$ ($X \succ Y$) if the matrix $X - Y$ is negative definite (respectively, positive) matrix. 0 and I represent compatibly dimensioned null and identity matrices, respectively. The transpose of a matrix A is denoted by A^T, and A^* denotes the complex conjugate transpose of a matrix. Also, the symbol $(\star)$ denotes transposed elements in a symmetric matrix. The symbol diag$\{X_1, X_2, \cdots, X_n\}$ denotes a block diagonal matrix with compatibly dimensioned diagonal blocks $X_1, X_2, \cdots, X_n$ and sym(Λ) reads for the matrix $\Lambda + \Lambda^T$. Finally, for a square matrix, say H, $\rho(H)$ denotes the spectral radius.

The following result is used in the analysis of this paper.

Lemma 1 *[2] Given a symmetric matrix $\Upsilon \in \mathbb{R}^{q \times q}$ and two matrices Λ, Σ of column dimension q, there exists a matrix W such that the LMI*

$$\Upsilon + \mathrm{sym}\left\{\Lambda^T W \Sigma\right\} \preceq 0,$$

holds if and only if the following two projection inequalities with respect to W are satisfied:

$$\Lambda^{\perp^T} \Upsilon \Lambda^{\perp} \preceq 0, \ \Sigma^{\perp^T} \Upsilon \Sigma^{\perp} \preceq 0, \tag{1}$$

where $\Lambda^{\perp}$ and $\Sigma^{\perp}$ are arbitrary matrices whose columns form a basis of the null spaces of Λ and Σ respectively.

2 Problem description

This paper considers ILC applied to a batch process, which receptively performs a given task over a finite duration $0 \leq t \leq \alpha$ (called a batch or a trial) and modeled as

$$
\begin{aligned}
\dot{x}_{k+1}(t) &= Ax_{k+1}(t) + A_d x_{k+1}(t-d) + Bu_{k+1}(t), \\
y_{k+1}(t) &= Cx_{k+1}(t),
\end{aligned}
\tag{2}
$$

where k denotes the discrete batch index and $\alpha < \infty$ is the batch duration. Also, $x_k(t) \in \mathbb{R}^n$, $y_k(t) \in \mathbb{R}^m$ and $u_k(p) \in \mathbb{R}^l$ are the state, output and input vectors on batch k. The time delay d is an unknown constant that satisfies $0 < d \leq \bar{d}$, where $\bar{d}$ is a known upper bound. Since the initial state of each batch is usually reset, it is assumed that $x_k(t) = x_{0,k}$, $t \in [-d, 0]$ on each batch.

Let $y_d(t)$, $0 \leq t \leq \alpha$, be a known reference trajectory. Then on batch k the error can be formed as

$$
e_k(t) = y_d(t) - y_k(t)
\tag{3}
$$

and the control objective is to construct a control law that forces convergence in k of the error sequence $\{e_k(t)\}_k$, i.e.,

$$
\lim_{k \to \infty} ||e_k(t)|| = 0, \ 0 \leq t \leq \alpha
\tag{4}
$$

where $|| \cdot ||$ is the norm on the underlying function space; in some practical applications, this condition must be weakened by requiring convergence within a suitably chosen (small) neighborhood of the origin. Moreover, it is also necessary to ensure acceptable dynamics in t.

The ILC law considered in this paper has the structure

$$
u_{k+1}(t) = u_k(t) + \Delta u_{k+1}(t),
\tag{5}
$$

i.e., the control used for the previous batch plus a correction $\Delta u_{k+1}(t)$ that can make use of previous trial data, where the particular case considered is

$$
\Delta u_{k+1}(t) = K_1 e_k(t) + K_2 \dot{e}_k(t) + K_3 e_k(t-d),
\tag{6}
$$

where K_1, K_2 and K_3 are gain matrices to be designed. Hence the control objective is to determine K_1, K_2 and K_3 of (6) such that the control input sequence generated by (5) over $0 \leq t \leq \alpha$ minimizes the tracking errors to achieve (4).

Remark 1 *The control law of (6) can be reduced to the standard PD-type ILC law when $K_3 = 0$. This assumption simplifies the design procedure and does not require complete knowledge of time delay d, see, e.g. [4].*

Remark 2 *The structure of (6) is simpler when compared to the alternative given in [6] since it does not include state feedback. Specifically, complete state measurement may be difficult to implement; hence, the new control law structure is better suited to physical applications.*

3 Repetitive Process based ILC design

Application of the control law (5)-(6) to (2) results in controlled batch process dynamics described by

$$
\begin{aligned}
\dot{x}_{k+1}(t) &= Ax_{k+1}(t) + A_d x_{k+1}(t-d) + BK_1 e_k(t) \\
&\quad + BK_2 \dot{e}_k(t) + BK_3 e_k(t-d) + Bu_k(t), \\
y_{k+1}(t) &= Cx_{k+1}(t).
\end{aligned}
\tag{7}
$$

The difficulty with this model is that there is no direct way of analyzing both batch-to-batch error convergence and the transient dynamics. This description of the dynamics is written as a member of the class of differential linear repetitive processes with delays to remove this difficulty. Hence, the interaction between transient dynamics and batch-to-batch error convergence can be investigated.

Next, to provide a basis for frequency domain design, the Laplace transform is applied to (7). Note that this transform requires input and output signals to be defined over an infinite time horizon and hence is not applicable due to the finite batch length. See, however, [1, 5] for the details of how to avoid the effects due to the finite batch length. Applying this transform (where, e.g., the Laplace transform of $x_{k+1}(t)$ is written as $X_{k+1}(s)$) gives

$$
\begin{aligned}
sX_{k+1}(s) &= AX_{k+1}(s) + \mathrm{e}^{-sd} A_d X_{k+1}(s) \\
&\quad + BK_1 E_k(s) + sBK_2 E_k(s) \\
&\quad + \mathrm{e}^{-sd} BK_3 E_k(s) + BU_k(s), \\
Y_{k+1}(s) &= CX_{k+1}(s).
\end{aligned} \tag{8}
$$

Consequently

$$
\begin{aligned}
(sI - A - \mathrm{e}^{-sd} A_d) X_{k+1}(s) &= B\left(K_1 + sK_2 + \mathrm{e}^{-sd} K_3\right) E_k(s) \\
&\quad + BU_k(s), \\
Y_{k+1}(s) &= CX_{k+1}(s)
\end{aligned} \tag{9}
$$

and in case of $U_k(s) = 0$

$$
\begin{aligned}
Y_{k+1}(s) &= C(sI - A - \mathrm{e}^{-sd} A_d)^{-1} \\
&\quad \times B(K_1 + sK_2 + \mathrm{e}^{-sd} K_3).
\end{aligned}
$$

Using the above results, the transfer function matrix coupling $e_k(t)$ to $y_{k+1}(t)$ is

$$
H(s) = G(s)[(K_1 + sK_2 + \mathrm{e}^{-sd} K_3) E_k(s)], \tag{10}
$$

where

$$
G(s) = C(sI - A - \mathrm{e}^{-sd} A_d)^{-1} B \tag{11}
$$

is the transfer function matrix coupling $u_{k+1}(t)$ to $y_{k+1}(t)$ (refer back to (7)). Next, define

$$
M(s, d) = sI - A - \mathrm{e}^{-sd} A_d
$$

and assume that $M(s, d)$ is nonsingular, then

$$
\begin{aligned}
M(s, d) M(s, d)^{-1} &= sM(s, d)^{-1} \\
&\quad - \left(A + \mathrm{e}^{-sd} A_d\right) M(s, d)^{-1} \\
&= I.
\end{aligned}
$$

Rewriting this last equation as

$$
sM(s, d)^{-1} = I + \left(A + \mathrm{e}^{-sd} A_d\right) M(s, d)^{-1}
$$

gives

$$H(s) = CM(s,d)^{-1}BK_1 + CBK_2 + CAM(s,d)^{-1}BK_2 \\ + e^{-sd}(CA_dM(s,d)^{-1}BK_2 + CM(s,d)^{-1}BK_3). \tag{12}$$

Next, focus on (9) again and assume that $E_k(s) = 0$. Then,

$$Y_{k+1}(s) = G(s)U_k(s),$$

where $G(s)$ is defined in (11). The next step is to consider the difference between successive batch errors since the propagation of the error from batch to batch must be analyzed, starting from

$$E_{k+1}(s) - E_k(s) = -G(s)(U_{k+1}(s) - U_k(s)). \tag{13}$$

Hence

$$E_{k+1}(s) = (I - H(s))E_k(s) \tag{14}$$

and therefore the tracking error converges as $k \to \infty$ if all eigenvalues of $I - H(s)$ have modulus less than unity, i.e.,

$$\rho\left(I - H\left(j\omega\right)\right) < 1, \forall \omega \in [0, \infty). \tag{15}$$

Also, the following condition in terms of matrix inequalities is the necessary and sufficient for (15) to hold

$$\left(I - H\left(j\omega\right)\right)^* P(j\omega)\left(I - H\left(j\omega\right)\right) - P(j\omega) \prec 0, \forall \omega \in [0, \infty), \tag{16}$$

where $P(j\omega)$ is a positive definite Hermitian matrix. Unfortunately, the dependence $P(j\omega)$ on ω is unknown, making the above inequality very hard to solve. Consequently, a constant P matrix over the entire frequency range is used, and the repetitive process setting given next is applied.

3.1   Repetitive process based analysis

To formulate the ILC design problem in the repetitive process setting, consider the term $I - H(s)$, which can be rewritten as

$$I - H(s) = G_1(s) + G_2(s) + G_3(s), \tag{17}$$

where

$$\begin{aligned} G_1(s) &= I - CBK_2 - CM(s,d)^{-1}BK_1, \\ G_2(s) &= -C\left(A + e^{-sd}A_d\right)M(s,d)^{-1}BK_2, \\ G_3(s) &= -Ce^{-sd}M(s,d)^{-1}BK_3. \end{aligned} \tag{18}$$

Hence the controlled system (7) can be equivalently converted to the following three differential repetitive processes in a parallel connection,

$$\begin{aligned} \dot{x}_{k+1}(t) &= Ax_{k+1}(t) + A_dx_{k+1}(t-d) + B_{0v}e_k(t), \\ e_{k+1}(t) &= C_vx_{k+1}(t) + C_{dv}x_{k+1}(t-d) + D_{0v}e_k(t). \end{aligned} \tag{19}$$

where the subscript v denotes for the repetitive process number, i.e. $v = \{1, 2, 3\}$. Note that the matrices A and A_d are the same for all three processes. The remaining matrices for the first process model $(v = 1)$ are

$$B_{01} = BK_1, \; C_1 = -C, \; C_{d1} = 0, D_{01} = (I - CBK_2), \tag{20}$$

for $v = 2$

$$B_{02} = BK_2, \ C_2 = -CA, \ C_{d2} = -CA_d, D_{02} = 0, \tag{21}$$

and for $v = 3$

$$B_{03} = BK_3, \ C_3 = 0, \ C_{d3} = -C, D_{03} = 0. \tag{22}$$

Given (15), the condition for tracking error convergence can be written as

$$\rho\left(G_1\left(j\omega\right) + G_2\left(j\omega\right) + G_3\left(j\omega\right)\right) < 1, \forall \omega \in [0, \infty). \tag{23}$$

However, even if (15) or (23) hold, poor transients in t occur even if convergence in k of the error sequence $\{e_k(t)\}_k$ in k occurs. (the dynamics in t occur over a finite duration where even an unstable linear system can only produce a bounded output in response to a bounded input.)

To avoid this problem, the condition in (15) is replaced by

$$\overline{\sigma}(I - H\left(j\omega\right)) < 1, \forall \omega \in [0, \infty), \tag{24}$$

where $\overline{\sigma}(\cdot)$ denotes the maximum singular value of its matrix argument, preventing the above problem from occurring. Moreover, if this last condition is, by definition

$$\|I - H\left(j\omega\right)\|_{\infty} \triangleq \max_{\omega \in [0,\infty)} \overline{\sigma}\left(I - H\left(j\omega\right)\right) < 1 \tag{25}$$

(This requires that the state matrix of the dynamics is stable). This is an $\mathcal{H}_{\infty}$ condition for batch-to-batch error convergence is formulated in $\mathcal{H}_{\infty}$ control setting. Moreover, (24) implies that

$$\|e_k(t)\|_2 \leq \|\left(I - H\left(j\omega\right)\right)\|_{\infty}^k \|e_0(t)\|_2, \tag{26}$$

where $\|\cdot\|_2$ denotes the L_2 norm, ensures monotonic batch-to-batch error convergence occurs.

The control synthesis problem is to design the control system represented by the closed-loop transfer function matrices $G_v(s)$, $v = 1, 2, 3$ such that

$$\begin{cases} \|G_1\left(j\omega\right)\|_{\infty} < 1 - \gamma_a - \gamma_b, \\ \|G_2\left(j\omega\right)\|_{\infty} < \gamma_a, \\ \|G_3\left(j\omega\right)\|_{\infty} < \gamma_b, \end{cases} \tag{27}$$

where γ_a and γ_b are given scalars satisfying $0 < \gamma_a < 1$ and $0 < \gamma_b < 1$. Then,

$$\begin{aligned} \|I - H\left(j\omega\right)\|_{\infty} &= \|G_1\left(j\omega\right) + G_2\left(j\omega\right) + G_3\left(j\omega\right)\|_{\infty} \\ &\leq \|G_1\left(j\omega\right)\|_{\infty} + \|G_2\left(j\omega\right)\|_{\infty} \\ &\quad + \|G_3\left(j\omega\right)\|_{\infty} < 1. \end{aligned}$$

Hence, when the system of inequalities

$$\begin{cases} \overline{\sigma}\left(G_1\left(j\omega\right)\right) < 1 - \gamma_a - \gamma_b, \\ \overline{\sigma}\left(G_2\left(j\omega\right)\right) < \gamma_a, \\ \overline{\sigma}\left(G_3\left(j\omega\right)\right) < \gamma_b \end{cases} \tag{28}$$

hold monotonic batch-to-batch error convergence is ensured. The design problem can now be formulated as: given the batch process model and scalars $0 < \gamma_a < 1$ and $0 < \gamma_b < 1$, determine

an ILC law of the form (5)-(6) such that the processes described by (19)-(22) are stable along the batch (trial).

By the triangle inequality, (27) can be rewritten as

$$\left\| G_v \left(j\omega \right) \right\|_\infty < \gamma_v, \tag{29}$$

for $v = 1, 2, 3$, where $\gamma_1 = 1 - \gamma_a - \gamma_b$, $\gamma_2 = \gamma_a$ and $\gamma_3 = \gamma_b$. The values of γ_a and γ_b must be selected based on knowledge of the particular example under consideration. The following theorem can now be established

Theorem 1 *Suppose that an ILC law of the form (6) is applied to systems described by (2) with specified reference trajectory $y_d(t)$ over $0 \le t \le \alpha$. Then the resulting controlled dynamics can be written in the form (19). Moreover, this last representation is stable along the batch for all delays $d \in \left[0, \overline{d} \right]$ if and only if*

   i) *$\rho(D_{0v}) < 1$,*

   ii) *all eigenvalues of the matrix $(A + e^{-j\omega d} A_d)$ have strictly negative real parts $\forall \omega \in [0, \infty)$ and $d \in \left[0, \overline{d} \right]$*

   iii) *the inequality (23) holds $\forall \omega \in [0, \infty)$ and $d \in \left[0, \overline{d} \right]$.*

The proof of this result follows from routine modifications to that for the corresponding result in [5] for $d = 0$ (no time delay) (where a batch is termed a trial). Hence the details are omitted. Moreover, the first condition guarantees convergence in k, also known as asymptotic stability. This condition places no constraint on the state matrix and hence on the dynamics in t. The obvious way to regulate the dynamics in t is to impose the second condition in the last result.

This second condition is not enough in all cases. An example to confirm this fact is given in [5] (for the delay-free case but extends naturally to the current case. Instead, the third condition in this last result, known as stability along the batch, is required. This condition that the frequency content of y_d must be attenuated at a geometric rate for all frequencies. Next, control law design is considered, resulting in LMI-based computations.

4   LMI-based design procedure

In this section, the main goal is to establish control law design algorithms. The route is based on a version of the KYP lemma and results in LMI-based computation of the control law matrices. The new results in this paper are based on a version KYP lemma, which can directly ensure the control performance specifications over the entire frequency range and any delay value.

Lemma 2 *Consider a differential linear repetitive process described by (19)-(22) with corresponding transfer-function matrix of (17). Suppose that scalars $\gamma_v \in (0, 1)$ for $v = 1, 2, 3$ are given. Then (16) holds if there exist Hermitian matrices $\mathcal{P}_v(j\omega) \succ 0$, $v = 1, 2, 3$, such that*

$$\begin{bmatrix} G_v(j\omega) \\ I \end{bmatrix}^* \begin{bmatrix} \mathcal{P}_v(j\omega) & 0 \\ 0 & -\gamma_v^2 \mathcal{P}_v(j\omega) \end{bmatrix} \begin{bmatrix} G_v(j\omega) \\ I \end{bmatrix} \prec 0, \tag{30}$$

$\forall \omega [0, \infty)$, where G_v, $v = 1, 2, 3$ are defined in (29).

The proof of this result follows by routine modification of that for the delay-free case, see [5], and hence the details are omitted. Additionally, (30) holds provided (28) or (29) are feasible for $\mathcal{P}_v(j\omega) = I$, $v = 1, 2, 3$.

The following result is a version of Theorem 1 of [8] to the single delay case over the entire frequency range ($\Psi = 0$). It gives an LMI-based sufficient condition, dependent on the upper bounds of time delay, for processes to satisfy specification (29).

Lemma 3 *Let real symmetric matrices Π_v of compatible dimensions and any delay d satisfying $0 < d \leq \bar{d}$ be given. Also, consider the differential linear repetitive processes described by (19)-(22) with corresponding transfer-function matrix (17). Then the condition of (29) is satisfied if there exist $P_v \succ 0$, $Z_v \succ 0$ and symmetric matrices X_v, such that*

$$
\begin{bmatrix} A & A_d & B_{0v} \\ I & 0 & 0 \end{bmatrix}^T \left(\Phi \otimes P_v + \Psi_0 \otimes \bar{d}Z_v \right) \begin{bmatrix} A & A_d & B_{0v} \\ I & 0 & 0 \end{bmatrix}
$$
$$
+ \begin{bmatrix} C_v & C_{dv} & D_{0v} \\ 0 & 0 & I \end{bmatrix}^T \Pi_v \begin{bmatrix} C_v & C_{dv} & D_{0v} \\ 0 & 0 & I \end{bmatrix} \tag{31}
$$
$$
+ \begin{bmatrix} X_v - \bar{d}^{-1}Z_v & \bar{d}^{-1}Z_v & 0 \\ \bar{d}^{-1}Z_v & -X_v - \bar{d}^{-1}Z_v & 0 \\ 0 & 0 & 0 \end{bmatrix} \prec 0,
$$

holds for $v = \{1, 2, 3\}$, where

$$
\Phi = \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}, \Psi_0 = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}.
$$

Moreover, the following frequency domain inequality holds for the transfer function matrices defined in (18)

$$
\begin{bmatrix} \sum_{v=1}^{3} G_v(j\omega) \\ I \end{bmatrix}^* \Pi_v \begin{bmatrix} \sum_{v=1}^{3} G_v(j\omega) \\ I \end{bmatrix} \prec 0, \forall \omega \in [0, \infty). \tag{32}
$$

Remark 3 *This last result requires that the matrices P and Z are positive definite to guarantee that all eigenvalues of the matrix $(A + e^{-j\omega d} A_d)$ have strictly negative real parts [8]. This requirement means that condition ii) of Lemma 1 is immediately satisfied.*

The inequality conditions given in Lemma 3 are not convex and, therefore, cannot be solved using the numerical solvers directly (e.g. MATLAB packages as LMI CONTROL TOOLBOX or SEDUMI). To obtain a convex problem, the following transformations are required.

Firstly, by Lemma 3, the choice of $\Pi_v = \begin{bmatrix} I & 0 \\ 0 & -\gamma_v^2 I \end{bmatrix}$ in (31) is made. Then (32) implies that the conditions of (28) are satisfied for entire frequency range, i.e. $[0, \infty)$. Next by defining

$$
\mathcal{M}_v = \begin{bmatrix} I & 0 \\ 0 & I \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \left(\Phi \otimes P_v + \Psi_0 \otimes \bar{d}Z_v \right) \begin{bmatrix} I & 0 \\ 0 & I \\ 0 & 0 \\ 0 & 0 \end{bmatrix}^T
$$
$$
+ \begin{bmatrix} 0 & 0 \\ C_v^T & 0 \\ C_{dv}^T & 0 \\ D_{0v}^T & I \end{bmatrix} \Pi_v \begin{bmatrix} 0 & 0 \\ C_v^T & 0 \\ C_{dv}^T & 0 \\ D_{0v}^T & I \end{bmatrix}^T + \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & X_v - \bar{d}^{-1}Z_v & \bar{d}^{-1}Z_v & 0 \\ 0 & \bar{d}^{-1}Z_v & -X_v - \bar{d}^{-1}Z_v & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \tag{33}
$$

it follows that the inequalities of (31) can be rewritten as

$$
\begin{bmatrix} A & A_d & B_{0v} \\ I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix}^T \mathcal{M}_v \begin{bmatrix} A & A_d & B_{0v} \\ I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix} \prec 0. \tag{34}
$$

Next, introduce positive scalars β_v and select

$$
\Lambda_v^\perp = \begin{bmatrix} A & A_d & B_{0v} \\ I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix}, \quad \Sigma_v^\perp = \begin{bmatrix} I & 0 & 0 \\ -\beta_v I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix},
$$

then by direct computation Λ_v and Σ_v are

$$
\Lambda_v = \begin{bmatrix} -I & A & A_d & B_{0v} \end{bmatrix}, \quad \Sigma_v = \begin{bmatrix} \beta_v I & I & 0 & 0 \end{bmatrix}. \tag{35}
$$

Also,

$$
\left(\Phi \otimes P_v + \Psi_0 \otimes \bar{d} Z_v \right) = \left[\begin{array}{c|c} \bar{d} Z_v & P_v \\ \hline P_v & 0 \end{array} \right] \tag{36}
$$

and

$$
\Sigma_v^{\perp T} \mathcal{M} \Sigma_v^\perp = \begin{bmatrix} \Sigma_{1v} & -\beta_v C_v^T C_{dv} - \beta_v \bar{d} Z_v & -\beta C_v^T D_{0v} \\ (\star) & C_{dv}^T C_{dv} - X_v - \bar{d}^{-1} Z_v & C_{dv}^T D_{0v} \\ (\star) & (\star) & D_{0v}^T D_{0v} - \gamma_v I \end{bmatrix}, \tag{37}
$$

where $\Sigma_{1v} = -\beta_v^2 (\bar{d} Z_v - X_v - C_v^T C_v) + \bar{d} Z_v$ for $v = 1, 2, 3$. Consequently, when the following inequalities (see (34) and (37))

$$
\Lambda_v^{\perp T} \mathcal{M}_v \Lambda_v^\perp \prec 0, \quad \Sigma_v^{\perp T} \mathcal{M}_v \Sigma_v^\perp \prec 0
$$

hold, Lemma 1 gives that there exists W_v such that

$$
\mathcal{M}_v + \mathrm{sym} \left\{ \Lambda_v^T W_v \Sigma_v \right\} \prec 0
$$

is feasible for $v = 1, 2, 3$. Since products exist among matrix variables, the last inequality is not in the LMI form. The next result solves this problem and reformulates the result of Lemma 3 into an LMI-based condition. This paper's main novel result is this theorem and enables control law design.

Theorem 2 *Suppose that a batch process of (2) is required to repeatedly follow the given reference $y_d(t)$ over $0 \le t \le \alpha$. Also, let ILC law (6) be applied and assume any delay d satisfying $0 < d \le \bar{d}$ is given. Additionally, let $\hat{S}_2$ be a given matrix and γ_v, β_v, be given scalars where $0 < \gamma_v \le 1$ (satisfying relations as in (29)), $\beta_v > 0$ for $v = 1, 2, 3$. Then the resulting dynamics can be represented by (19)-(22) with corresponding transfer-function matrix of (17). Moreover the*

condition of (29) is satisfied if there exist $\hat{P}_v \succ 0$, $\hat{Z}_v \succ 0$, $\hat{S}_1 \succ 0$, $\hat{S}_3 \succ 0$ symmetric matrices X_v, N_1, N_2 and N_3 together with positive scalars $\gamma_v > 0$ such that

$$
\left[
\begin{array}{cc}
\bar{d}\hat{Z}_v - \mathrm{sym}\{\beta_v \hat{S}_v\} & \beta_v A \hat{S}_v - \hat{S}_v^T + \hat{P}_v \\
(\star) & \mathrm{sym}\{A\hat{S}_v\} - \bar{d}^{-1}\hat{Z}_v + \hat{X}_v \\
(\star) & (\star) \\
(\star) & (\star) \\
(\star) & (\star)
\end{array}
\right.
$$
$$
\left.
\begin{array}{ccc}
\beta_v A_d \hat{S}_v & \beta_v B N_v & 0 \\
A_d \hat{S}_v + \bar{d}^{-1}\hat{Z}_v & B N_v & \hat{S}_v^T C_v^T \\
-\hat{X}_v - \bar{d}^{-1}\hat{Z}_v & 0 & \hat{S}_v^T C_{dv}^T \\
(\star) & -\gamma_v^2 I & D_{0v}^T \\
(\star) & 0 & -I
\end{array}
\right] \prec 0.
\tag{38}
$$

hold for $v = \{1, 2, 3\}$. Moreover, if these LMIs are feasible, the corresponding matrices gain matrices K_1, K_2, and K_3 in updating law (6) can be obtained from

$$
K_1 = N_1 \hat{S}_1^{-1}, \ K_2 = N_2 \hat{S}_2^{-1}, K_3 = N_3 \hat{S}_3^{-1}.
\tag{39}
$$

Proof 1 *Suppose that the LMI (38) is feasible. Then pre- and post-multiply (38) by* $\mathrm{diag}\left\{S_v^{-T}, S_v^{-T}, S_v^{-T}, I, I\right\}$ *and its transpose to obtain*

$$
\left[
\begin{array}{cc}
\bar{d}\hat{S}_v^{-T}\hat{Z}_v \hat{S}_v^{-1} - \mathrm{sym}\{\beta_v \hat{S}_v^{-T}\} & \Upsilon_{12} \\
(\star) & \Upsilon_{13} \\
(\star) & (\star) \\
(\star) & (\star) \\
(\star) & (\star)
\end{array}
\right.
$$
$$
\left.
\begin{array}{ccc}
\beta_v \hat{S}_v^{-T} A_d & \beta_v \hat{S}_v^{-T} B_{0v} & 0 \\
\hat{S}_v^{-T} A_d + \bar{d}^{-1}\hat{S}_v^{-T}\hat{Z}_v \hat{S}_v^{-1} & \hat{S}_v^{-T} B_{0v} & C_v^T \\
0 & 0 & C_{dv}^T \\
(\star) & -\gamma_v^2 I & D_{0v}^T \\
(\star) & 0 & -I
\end{array}
\right] \prec 0.
\tag{40}
$$

where

$$
\Upsilon_{12} = \beta_v \hat{S}_v^{-T} A - \hat{S}_v^{-1} + \hat{S}_v^{-T}\hat{P}\hat{S}_v^{-1},
$$
$$
\Upsilon_{13} = \mathrm{sym}\{\hat{S}_v^{-T} A\} - \bar{d}^{-1}\hat{S}_v^{-T}\hat{Z}_v \hat{S}_v^{-1} + \hat{S}_v^{-T}\hat{X}_v \hat{S}_v^{-1}.
$$

Next introduce the following change of variables as in (39) and

$$
S_v^{-1} = W_v, X_v = S_v^{-T}\hat{X}_v S_v^{-1}, Z_v = S_v^{-T}\hat{Z}_v S_v^{-1},
$$
$$
P_v = S_v^{-T}\hat{P}_v S_v^{-1}.
$$

Application of Schur's complement formula gives that (40) *holds if and only if*

$$
\begin{bmatrix}
-\operatorname{sym}\{\beta_v W_v\} & \beta_v W_v^T A - W_v & \beta_v W_v^T A_d & \beta_v W_v^T B_{0v} \\
(\star) & \operatorname{sym}\{A^T W_v\} & W_v^T A_d & W_v^T B_{0v} \\
(\star) & (\star) & 0 & 0 \\
(\star) & (\star) & (\star) & 0
\end{bmatrix}
$$
$$
+ \begin{bmatrix}
\bar{d} Z_v & P_v & 0 & 0 \\
(\star) & \Upsilon_{22} & C_v^T C_{dv} + \bar{d}^{-1} Z_v & C_{dv}^T D_{0v} \\
(\star) & (\star) & \Upsilon_{33} & C_{dv}^T D_{0v} \\
(\star) & (\star) & (\star) & D_{0v}^T D_{0v} - \gamma_v I
\end{bmatrix} \prec 0
$$

holds where

$$
\Upsilon_{22} = C_v^T C_v - \bar{d}^{-1} Z_v + X_v, \ \Upsilon_{33} = -X_v - \bar{d}^{-1} Z_v + C_{dv}^T C_{dv}.
$$

Moreover, the above inequalities can be rewritten as

$$
\mathcal{M}_v + \operatorname{sym}\left\{\Lambda_v^T W \Sigma_v\right\} \prec 0, \tag{41}
$$

where $\mathcal{M}_v$, Λ_v and Σ_v are defined in (33) *and* (35) *respectively. Then, given Lemma 1 it follows that* (41) *is feasible if and only if* (31) *hold and the proof is complete.*

Remark 4 *The matrix $\hat{S}_2$ in* (38) *cannot be declared as the matrix variable since the term D_{01} (for $v = 1$) includes K_2 (or N_2) and this controller gain matrix is not multiplied by any matrix variable. Hence there is no possibility to introduce any (linearizing) change of variables.*

5   Case study

To illustrate the new design procedure, the results of an numerical simulation on a two-stage chemical reactor with delayed recycle streams are presented and discussed. Both of the reactors are isothermal continuous stirred tank reactors (CSTR). A reactor recycle does not increase the overall conversion and reduces the cost of a reaction and therefore it is commonly used in industrial applications. The input to be recycled must be separated from the yields and then travel through pipes. The total recycle time therefore introduces delays in the state vector.

Following the detailed description in [6] the resulting batch process model is of the form (2) with

$$
A = \begin{bmatrix} -2.5 & 0 \\ 1 & -2.5 \end{bmatrix}, A_d = \begin{bmatrix} 0 & 0.4 \\ 0 & 0 \end{bmatrix}, B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, C = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.
$$

Also, the time-delay range is $0 < d \leq \bar{d} = 1$, where $d = 1$ is used in the simulation results. The state initial vector $x_k(0)$ and the input vector $u_k(0)$ are assumed to be zero $\forall k \geq 0$ and the reference trajectories are (they are depicted in Fig. 1):

$$
y_{1d}(t) = \begin{cases}
\frac{1}{20}t, & 0 \leq t < 40, \\
2 + \frac{1}{120}(t - 40), & 40 \leq t < 100, \\
2.5 + \frac{1}{300}(t - 100), & 100 \leq t < 250, \\
3 & 250 \leq t \leq 300,
\end{cases}
$$

$$y_{2d}(t) = \begin{cases} \frac{1}{120}t, & 0 \le t < 60 \\ 0.5 + \frac{1}{60}(t-60), & 60 \le t < 120 \\ 1.5, & 120 \le t < 150 \\ 1.5 + \frac{1}{100}(t-150), & 150 \le t < 250 \\ 2.5, & 250 \le t \le 300 \end{cases},$$

To evaluate tracking performance from batch to batch, let e_{ik}, $i = 1, 2$ denote the tracking errors of output i and batch k. Then the convergence measure is the root mean square (RMS):

$$\mathrm{RMS}(ik) = \frac{1}{300} \int_0^{300} (y_{id}(t) - y_{ik}(t))^2 \, \mathrm{d}t,$$

where the smaller the value of this quantity the better the tracking performance along the kth batch.
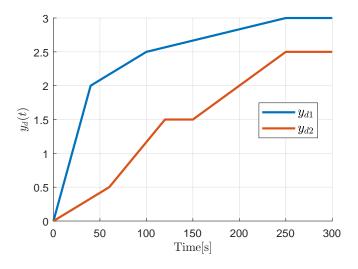

Figure 1: The reference trajectories.

By applying the design procedure given in Theorem 2 for $\beta_1 = 0.1$, $\beta_2 = 0.1$, $\beta_3 = 1$, $\gamma_1 = 0.499$, $\gamma_2 = 0.3$, $\gamma_3 = 0.2$ and $\hat{S}_2 = \mathrm{diag}\{0.5, 0.5\}$ the controller matrices are

$$K_1 = \begin{bmatrix} 0.6680 & 0.1018 \\ 0.0745 & 0.6550 \end{bmatrix}, \quad K_2 = \begin{bmatrix} 0.3226 & 0.0116 \\ 0.0149 & 0.3389 \end{bmatrix},$$

$$K_3 = 10^{-17} \cdot \begin{bmatrix} 0.0789 & -0.0472 \\ -0.0278 & 0.1332 \end{bmatrix}.$$

In this case, $K_3 \approx 0$ has little influence on the resulting performance (tracking error performance). The simulation results are shown in Fig. 2, where the tracking effectiveness of the ILC approach is assessed using the RMSE (Root Mean Square Error) of output tracking errors for the first 40 batches and confirms the tracking errors' convergence. Furthermore, from Figure 2, it is seen that the developed design procedure yields rapid tracking error reduction from batch to batch, which is comparable with the design in [6]. This new result does not use state feedback and finite frequency ranges.
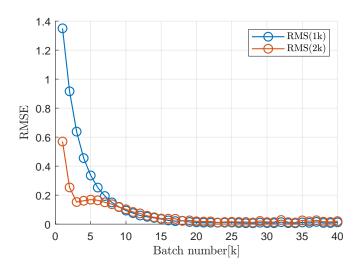
Figure 2: The RMSE against batch number

6 Conclusion

This paper has developed a new procedure for designing ILC laws for differential batch processes with time delays. The new design for PD-type control laws has been developed using the repetitive process setting and the KYP lemma. Then, sufficient conditions for the convergence of the batch-to-batch error have been established in terms of LMIs. Finally, a numerical example demonstrates the effectiveness of the new design. Future work will address the inclusion of multiple delays and uncertainty in the ILC design. Extending the theory to include disturbance attenuation is another possible topic for future work. Also, adding the output feedback controller is another area for possible future research.

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