

## **SURFACE WAVES IN FIBRE-REINFORCED ANISOTROPIC SOLID ELASTIC MEDIA UNDER THE INFLUENCE OF GRAVITY**

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The aim of the present paper is to investigate surface waves in an anisotropic, elastic solid medium under the influence of gravity. First, a theory of generalised surface waves was developed and then it was employed to investigate particular cases of waves, viz., Stoneley and Rayleigh, Love type. The wave velocity equations were obtained for different cases and they are in well agreement with the corresponding classical result, when the effect of gravity, viscosity as well as parameters for fibre-reinforcement of the material medium are ignored.

**Key words:** fibre-reinforced medium, surface waves, viscosity, Rayleigh waves, gravity.

### **1. Introduction**

The propagation of surface waves in homogeneous and non-homogeneous elastic media is of considerable importance in earth-quake engineering and seismology on account of occurrence of non-homogeneities in the earth crust, as the earth is made up of different layers. As a result, the theory of surface waves was developed by Stoneley (1924), Bullen (1965), Ewing *et al.* (1957), Hunters (1970) and Jeffreys (1970).

The effect of gravity on wave propagation in an elastic solid medium was first considered by Bromwich (1898), who treated the force of gravity as a type of body force. Love (1965) extended the work of Bromwich and investigated the influence of gravity on superfacial waves and showed that the Rayleigh wave velocity is affected by the gravity field. Sezawa (1927) studied the dispersion of elastic waves propagated on curved surfaces.

The transmission of elastic waves through a stratified solid medium was studied by Thomson. Haskell (1953) studied the dispersion of surface waves in multilayered media. The monograph of Ewing *et al.* (1957) is a comprehensive study of elastic waves.

Biot (1965) studied the influence of gravity on Rayleigh waves, assuming the force of gravity to create a type of initial stress of a hydrostatic nature and the medium to be incompressible. Taking into

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account the effect of initial stresses and using Biot's theory of incremental deformations, Dey modified the work of Jones (1964). De and Sengupta (1974) studied many problems of elastic waves and vibrations under the influence of gravity field. Sengupta and Acharya (1979) studied the influence of gravity on the propagation of waves in a thermoelastic layer. Brunelle (1973) studied the surface wave propagation under initial tension or compression. Wave propagation in a thin two-layered laminated medium with stress couples under initial stresses was studied by Roy (1984). Roy and Sengupta investigated the rotatory vibration of a general viscoelastic solid sphere and also the radial vibration of a general viscoelastic solid sphere. The details can be found in the work of Eringen and Sahubi (1975). Datta (1986) studied the effect of gravity on the Rayleigh wave propagation in a homogeneous, isotropic elastic solid medium. Goda (1992) studied the effect of inhomogeneity and anisotropy on Stoneley waves. Recently Abd-Alla and Ahmed (1996) studied the Rayleigh waves in an orthotropic thermoelastic medium under gravity field and initial stress.

In most previous investigations, the effect of reinforcement has been neglected. The idea of continuous self-reinforcement at every point of an elastic solid was introduced by Belfield *et al.* (1983). The characteristic property of a reinforced concrete member is that its components, namely concrete and steel, act together as a single anisotropic unit as long as they remain in the elastic condition, i.e., the two components are bound together so that there can be no relative displacement between them.

In this paper, the authors study the propagation of surface waves in anisotropic, fibre-reinforced solid media under the influence of gravity. Biot's theory of incremental deformations is used to obtain the wave velocity equation for Stoneley, Rayleigh and Love waves. These equations are in complete agreement with the corresponding classical results in the absence of gravity and fibre-reinforced parameters of the material medium.

## 2. Formulation of the problem

Let  $M_1$  and  $M_2$  be two fibre-reinforced elastic anisotropic semi-infinite solid media. They are perfectly welded in contact to prevent any relative motion or sliding before and after the disturbances and to ensure that the continuity of displacement, stress etc. hold good across the common boundary surface. The mechanical properties of  $M_1$  are different from those of  $M_2$ . These media extend to an infinite great distance from the origin and are separated by a plane horizontal boundary and  $M_2$  is to be taken above  $M_1$ .

Let  $Oxyz$  be a set of orthogonal Cartesian co-ordinates and let  $O$  be any point on the plane boundary and  $Oz$  points vertically downward to the medium  $M_1$ . We consider the possibility of a type of wave travelling in the direction  $Ox$ , in such a manner that the disturbance is largely confined to the neighbourhood of the boundary and at any instant, all particles in any line parallel to the  $y$ -axis have equal displacements. It follows from these two assumptions that the wave is a surface wave and all partial derivatives with respect to  $y$  are zero.

Further let us assume that  $u, v, w$  are the components of displacements at any point  $(x, y, z)$  at any time  $t$ . It is also assumed that the gravitational field generates a hydrostatic initial stress which is produced by a slow process of creep where the shearing stresses tend to become small or vanish after a long period of time.

The equilibrium equation of the initial stress is in the form

$$\frac{\partial \tau}{\partial x} = 0, \quad \frac{\partial \tau}{\partial z} + \rho g = 0.$$

The dynamical equations of motion for a three-dimensional elastic solid medium under the influence of initial stress and gravity (Biot 1965, pp. 44-45, 273-281) are

$$\frac{\partial \tau_{11}}{\partial x} + \frac{\partial \tau_{12}}{\partial y} + \frac{\partial \tau_{13}}{\partial z} + \rho g \frac{\partial w}{\partial x} = \rho \frac{\partial^2 u}{\partial t^2}, \quad (2.1a)$$

$$\frac{\partial \tau_{12}}{\partial x} + \frac{\partial \tau_{22}}{\partial y} + \frac{\partial \tau_{23}}{\partial z} + \rho g \frac{\partial w}{\partial y} = \rho \frac{\partial^2 v}{\partial t^2}, \quad (2.1b)$$

$$\frac{\partial \tau_{13}}{\partial x} + \frac{\partial \tau_{23}}{\partial y} + \frac{\partial \tau_{33}}{\partial z} - \rho g \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) = \rho \frac{\partial^2 w}{\partial t^2} \quad (2.1c)$$

where  $\rho$  is the density of the material medium,  $g$  is the acceleration due to gravity and  $\tau_{ij} = \tau_{ji} \forall i, j$  are the stress components.

The constitutive equations for a fibre-reinforced linearly elastic anisotropic medium with respect to a preferred direction  $\mathbf{a}$  are (Belfield *et al.*, 1983)

$$\begin{aligned} \tau_{ij} = & \lambda e_{kk} \delta_{ij} + 2\mu_T e_{ij} + \alpha (a_k a_m e_{km} \delta_{ij} + e_{kk} a_i a_j) + \\ & + 2(\mu_L - \mu_T)(a_i a_k e_{kj} + a_j a_k e_{ki}) + \beta (a_k a_m e_{km} a_i a_j) \end{aligned} \quad (2.2)$$

where  $e_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i})$  are components of strain;  $\alpha, \beta, (\mu_L - \mu_T)$  are reinforced anisotropic elastic parameters;  $\lambda, \mu_T$  are elastic parameters;  $\mathbf{a} = (a_1, a_2, a_3)$ ,  $a_1^2 + a_2^2 + a_3^2 = 1$ . If  $\mathbf{a}$  has components that are  $(1, 0, 0)$  so that the preferred direction is the  $x$ -axis, (2) simplifies, as given below

$$\begin{aligned} \tau_{11} = & (\lambda + 2\alpha + 4\mu_L - 2\mu_T + \beta)e_{11} + (\lambda + \alpha)e_{22} + (\lambda + \alpha)e_{33}, \\ \tau_{22} = & (\lambda + \alpha)e_{11} + (\lambda + 2\mu_T)e_{22} + \lambda e_{33}, \\ \tau_{33} = & (\lambda + \alpha)e_{11} + \lambda e_{22} + (\lambda + 2\mu_T)e_{33}, \\ \tau_{23} = & 2\mu_T e_{23}, \\ \tau_{13} = & 2\mu_L e_{13}, \\ \tau_{12} = & 2\mu_L e_{12}. \end{aligned} \quad (2.3)$$

Introducing Eqs (2.2), (2.3) in (2.1a), (2.1b), (2.1c), the equations of motion become

$$(\lambda + 2\alpha + 4\mu_L - 2\mu_T + \beta) \frac{\partial^2 u}{\partial x^2} + (\alpha + \lambda + \mu_L) \frac{\partial^2 w}{\partial x \partial z} + \mu_L \frac{\partial^2 u}{\partial z^2} + \rho g \frac{\partial w}{\partial x} = \rho \frac{\partial^2 u}{\partial t^2}, \quad (2.4)$$

$$(\mu_L - \mu_T) \frac{\partial^2 v}{\partial x^2} + \mu_T \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2} \right) v = \rho \frac{\partial^2 v}{\partial t^2}, \quad (2.5)$$

$$\mu_L \frac{\partial^2 w}{\partial x^2} + (\alpha + \lambda + \mu_L) \frac{\partial^2 u}{\partial x \partial z} + (\lambda + 2\mu_T) \frac{\partial^2 w}{\partial z^2} + \rho g \frac{\partial w}{\partial y} - \rho g \left( \frac{\partial u}{\partial x} \right) = \rho \frac{\partial^2 w}{\partial t^2}. \quad (2.6)$$

To examine dilatational and rotational disturbances, we introduce two displacement potentials  $\phi$  and  $\psi$  by the relations

$$u = \frac{\partial\phi}{\partial x} - \frac{\partial\psi}{\partial z}, \quad \omega = \frac{\partial\phi}{\partial z} + \frac{\partial\psi}{\partial x}. \quad (2.7)$$

The component  $v$  is associated with a purely distortional movement. We note that  $\phi$ ,  $\psi$  and  $v$  are associated with  $P$  waves,  $SV$  waves and  $SH$  waves, respectively. The symbols have their usual significance. Now using Eq.(2.7) in Eqs (2.4), (2.5) we obtain the following wave equation in  $M_I$  satisfied by  $\phi$  and  $\psi$  as

$$(\lambda + 2\alpha + 4\mu_L - 2\mu_T + \beta) \frac{\partial^2\phi}{\partial x^2} + (\alpha + \lambda + 2\mu_L) \frac{\partial^2\phi}{\partial z^2} + \rho g \frac{\partial\psi}{\partial x} = \frac{\partial^2\phi}{\partial t^2}, \quad (2.8)$$

$$(\alpha + 3\mu_L + \beta - 2\mu_T) \frac{\partial^2\psi}{\partial x^2} + \mu_L \frac{\partial^2\psi}{\partial z^2} - \rho g \frac{\partial\phi}{\partial x} = \frac{\partial^2\psi}{\partial t^2}, \quad (2.9)$$

$$(\mu_L - \mu_T) \frac{\partial^2 v}{\partial x^2} + \mu_T \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2} \right) v = \rho \frac{\partial^2 v}{\partial t^2}, \quad (2.10)$$

and similar relations in  $M_2$  with  $\rho, \lambda, \alpha, \mu_L, \beta$  replaced by  $\rho', \lambda', \alpha', \mu'_L, \beta'$ .

### 3. Solution of the problem

Now our main objective is to solve Eqs (2.8), (2.9), (2.10). We seek the solution of Eqs (2.8), (2.9), (2.10) in the following forms

$$(\phi, \psi, v) = [F(z), G(z), H(z)] e^{i\omega(x-ct)}. \quad (3.1)$$

Using Eq.(3.1) in Eqs (2.8), (2.9), and (2.10) we get a set of differential equations for the medium  $M_I$  as follows

$$\begin{aligned} \frac{d^2 F}{dz^2} + h_I^2 F + f_I^2 G &= 0, \\ \frac{d^2 G}{dz^2} + l_I^2 G - m_I^2 F &= 0, \\ \frac{d^2 H}{dz^2} + K_I^2 H &= 0 \end{aligned} \quad (3.2)$$

where 
$$h_I^2 = \frac{\omega^2 (C^2 - A_I)}{A_2}; \quad f_I^2 = \frac{gI\omega}{A_2},$$

$$l_1^2 = \frac{\omega^2 (C^2 - A_3)}{A_4}; \quad m_1^2 = \frac{i\omega g}{A_4}; \quad K_1^2 = \frac{\omega^2 (C^2 - A_4)}{A_5},$$

$$A_1 = \frac{\lambda + 2\alpha + 4\mu_L - 2\mu_T + \beta}{\rho}, \quad A_2 = \frac{\lambda + \alpha + 2\mu_L}{\rho}, \quad A_3 = \frac{\alpha + 3\mu_L + \beta - 2\mu_T}{\rho},$$

$$A_4 = \frac{\mu_L}{\rho}, \quad A_5 = \frac{\mu_T}{\rho},$$

and those for the medium  $M_2$  are given by

$$\frac{d^2 F}{dz^2} + h_1'^2 F + f_1'^2 G = 0,$$

$$\frac{d^2 G}{dz^2} + l_1'^2 G - m_1'^2 F = 0, \quad (3.3)$$

$$\frac{d^2 H}{dz^2} + K_1'^2 H = 0$$

where

$$h_1'^2 = \frac{\omega^2 (C^2 - A_1')}{A_2'}, \quad f_1'^2 = \frac{gi\omega}{A_2'}, \quad l_1'^2 = \frac{\omega^2 (C^2 - A_3')}{A_4'},$$

$$m_1'^2 = \frac{i\omega g}{A_4'}, \quad K_1'^2 = \frac{\omega^2 (C^2 - A_4')}{A_5'},$$

$$A_1' = \frac{\lambda' + 2\alpha' + 4\mu_L' - 2\mu_T' + \beta'}{\rho'}, \quad A_2' = \frac{\lambda' + \alpha' + 2\mu_L'}{\rho'}, \quad A_3' = \frac{\alpha' + 3\mu_L' + \beta' - 2\mu_T'}{\rho'},$$

$$A_4' = \frac{\mu_L'}{\rho'}, \quad A_5' = \frac{\mu_T'}{\rho'}.$$

Equations (3.2) and (3.3) must have exponential solutions in order that  $F, G, H$  will describe surface waves, they must become vanishingly small as  $z \rightarrow \infty$ .

Hence for the medium  $M_1$

$$\phi(x, z, t) = [Ae^{-p_1 z} + Be^{-p_2 z}] e^{i\omega(x-ct)},$$

$$\psi(x, z, t) = [Ce^{-p_3 z} + De^{-p_2 z}] e^{i\omega(x-ct)}, \quad (3.4)$$

$$v(x, z, t) = Ee^{-K_1 z + i\omega(x-ct)},$$

and for the medium  $M_2$

$$\begin{aligned}\phi(x, z, t) &= [A'e^{-p_1'z} + B'e^{-p_2'z}]e^{i\omega(x-ct)}, \\ \psi(x, z, t) &= [C'e^{-p_1'z} + D'e^{p_2'z}]e^{i\omega(x-ct)}, \\ v(x, z, t) &= E'e^{-K_1'z+i\omega(x-ct)}\end{aligned}\quad (3.5)$$

where  $p_j, p_j'$  ( $j=1, 2$ ) are roots of the equations

$$p^4 + (l_1^2 + h_1^2)p^2 + h_1^2l_1^2 + m_1^2f_1^2 = 0, \quad (3.6)$$

and 
$$p'^4 + (l_1'^2 + h_1'^2)p'^2 + h_1'^2l_1'^2 + m_1'^2f_1'^2 = 0. \quad (3.7)$$

For the media  $M_1$  and  $M_2$  respectively, we take into consideration the real roots of Eq.(3.6) and Eq.(3.7). The constants  $A, B$  and  $A', B'$  are related with  $C, D$  and  $C', D'$  in Eqs (3.4) and (3.5) by means of first Eqs in (3.2) and (3.3).

Equating the co-efficients of  $e^{-p_1z}, e^{-p_2z}, e^{p_1z}, e^{p_2z}$  to zero after substituting Eqs (3.4) and (3.5) in the first Eqs in (3.2) and (3.3) respectively, we get

$$C = \gamma_1 A, \quad D = \gamma_2 B, \quad C' = \gamma_1' A', \quad D' = \gamma_1' B'$$

where 
$$\gamma_j = -\frac{[p_j^2 + h_j^2]}{f_j^2} \quad \text{and} \quad \gamma_j' = -\frac{[p_j'^2 + h_j'^2]}{f_j'^2} \quad [j = 1, 2]. \quad (3.8)$$

#### 4. Boundary conditions

The boundary conditions for the problem are:

(i) The displacement components at the boundary surface between the media  $M_1$  and  $M_2$  must be continuous at all times and positions.

i.e., 
$$[u, v, w]M_1 = [u, v, w]M_2 \quad \text{at} \quad z=0 \quad \text{respectively.}$$

(ii) The stress components  $\tau_{31}, \tau_{32}$  and  $\tau_{33}$  must be continuous at the boundary  $z=0$ .

i.e., 
$$[\tau_{31}, \tau_{32}, \tau_{33}]M_1 = [\tau_{31}, \tau_{32}, \tau_{33}]M_2 \quad \text{at} \quad z=0 \quad \text{respectively,}$$

where

$$\tau_{31} = \mu_L \left( 2 \frac{\partial^2 \phi}{\partial x \partial z} + \frac{\partial^2 \psi}{\partial x^2} - \frac{\partial^2 \psi}{\partial z^2} \right),$$

$$\tau_{33} = \lambda \nabla^2 \phi + \alpha \left( \frac{\partial^2 \phi}{\partial x^2} - \frac{\partial^2 \psi}{\partial x \partial z} \right) + 2\mu_T \left( \frac{\partial^2 \phi}{\partial z^2} + \frac{\partial^2 \psi}{\partial x \partial z} \right), \quad (4.1)$$

$$\tau_{32} = \mu_T \frac{\partial v}{\partial z}$$

where  $\nabla^2$  is the two dimensional Laplacian operator given by

$$\nabla^2 \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2}.$$

Applying the boundary conditions (i) and (ii), we have

$$A(I - i\beta_1\gamma_1) + B(I - i\beta_2\gamma_2) - A'(i\beta'_1\gamma'_1 + I) - B'(i\beta'_2\gamma'_2 + I) = 0, \quad (4.2a)$$

$$E = E', \quad (4.2b)$$

$$A(\gamma_1 + i\beta_1) + B(\gamma_2 + i\beta_2) + A'(-\gamma'_1 + i\beta'_1) + B'(-\gamma'_2 + i\beta'_2) = 0, \quad (4.2c)$$

$$\begin{aligned} \mu_L \left[ (2i\beta_1 + \gamma_1 + \beta_1^2\gamma_1)A + (2i\beta_2 + \gamma_2 + \beta_2^2\gamma_2)B \right] = \\ = \mu'_L \left[ (-2i\beta'_1 + \gamma'_1 + \beta_1'^2\gamma'_1)A' + (-2i\beta'_2 + \gamma'_2 + \beta_2'^2\gamma'_2)B' \right], \end{aligned} \quad (4.2d)$$

$$\mu_T [-K_1 E] = \mu'_T [-K'_1 E'], \quad (4.2e)$$

$$\begin{aligned} A \left[ \lambda(\beta_1^2 - I) + 2\mu_T(\beta_1^2 - i\beta_1\gamma_1) + \alpha(-I + i\beta_1\gamma_1) \right] + \\ + B \left[ \lambda(\beta_2^2 - I) + 2\mu_T(\beta_2^2 - i\beta_2\gamma_2) + \alpha(-I + i\beta_2\gamma_2) \right] = \\ = A' \left[ \lambda'(\beta_1'^2 - I) + 2\mu'_T(\beta_1'^2 - i\beta_1'\gamma'_1) + \alpha'(-I - i\beta_1'\gamma'_1) \right] + \\ + B' \left[ \lambda'(\beta_2'^2 - I) + 2\mu'_T(\beta_2'^2 - i\beta_2'\gamma'_2) + \alpha'(-I - i\beta_2'\gamma'_2) \right] \end{aligned} \quad (4.2f)$$

where

$$\beta_j = \frac{p_j}{\omega}, \quad \beta'_j = \frac{p'_j}{\omega}, \quad j = 1, 2.$$

From Eqs (4.2b) and (4.2e), we have  $E = E' = 0$ . Thus there is no propagation of displacement  $v$ . Hence SH-waves are decoupled in this case.

Finally, eliminating the constants  $A, B, A', B'$  from Eqs (4.2a), (4.2c), (4.2d) and (4.2f), we get

$$\det(a_{ij}) = 0, \quad i, j = 1, 2, 3, 4 \quad (4.3)$$

where

$$\begin{aligned}
a_{11} &= (I - i\beta_1\gamma_1), & a_{12} &= I - i\beta_2\gamma_2, & a_{13} &= -(i\beta_1'\gamma_1' + I), & a_{14} &= -(i\beta_2'\gamma_2' + I), \\
a_{21} &= \gamma_1 + i\beta_1, & a_{22} &= \gamma_2 + i\beta_2, & a_{23} &= -\gamma_1' + i\beta_1', & a_{24} &= -\gamma_2' + i\beta_2', \\
a_{31} &= \mu_L \left[ (2i\beta_1 + \gamma_1 + \beta_1^2\gamma_1) \right], & a_{32} &= \mu_L \left[ 2i\beta_2 + \gamma_2 (I + \beta_2^2) \right] \mu_L, \\
a_{33} &= - \left[ -2i\beta_1' + \gamma_1' (I + \beta_1'^2) \right] \mu_L', & a_{34} &= - \left[ -2i\beta_2' + \gamma_2' (I + \beta_2'^2) \right] \mu_L', \\
a_{41} &= \left[ \lambda (\beta_1^2 - I) + 2\mu_T (\beta_1^2 - i\beta_1\gamma_1) + \alpha (-I + i\beta_1\gamma_1) \right], \\
a_{42} &= \left[ \lambda (\beta_2^2 - I) + 2\mu_T (\beta_2^2 - i\beta_2\gamma_2) + \alpha (-I + i\beta_2\gamma_2) \right], \\
a_{43} &= - \left[ \lambda' (\beta_1'^2 - I) + 2\mu_T' (\beta_1'^2 - i\beta_1'\gamma_1') + \alpha' (-I - i\beta_1'\gamma_1') \right], \\
a_{44} &= \left[ \lambda' (\beta_2'^2 - I) + 2\mu_T' (\beta_2'^2 - i\beta_2'\gamma_2') + \alpha' (-I - i\beta_2'\gamma_2') \right].
\end{aligned} \tag{4.4}$$

From Eq.(4.3), we get the velocity of surface waves in common boundary between two fibre-reinforced elastic anisotropic semi-infinite solid media under the influence of gravity. Since the wave velocity  $c$  obtained from Eq.(4.3) depends on the particular value of  $\omega$  which indicates the dispersion of the general wave form and on the gravity field.

## 5. Particular cases

**Stoneley waves:** are the generalised form of Rayleigh waves in which we assume that the waves are propagated along the common boundary of two semi-infinite media  $M_1$  and  $M_2$ . Therefore Eq.(4.3) determines the wave velocity equation for Stoneley waves in anisotropic fibre-reinforced solid elastic media under the influence of gravity.

Clearly, from Eq.(4.3) it is follows that wave velocity of Stoneley waves depends upon the parameters for fibre-reinforcement of the material medium, gravity and the densities of both media. The wave velocity Eq.(4.3) for Stoneley waves under the present circumstances depends on the particular value of  $\omega$  and creates a dispersion of a general wave form.

Equation (4.3), of course, is in complete agreement with the corresponding classical result, when the effect of gravity and parameters for fibre-reinforcement are ignored.

**Rayleigh waves:** To investigate the possibility of Rayleigh waves in anisotropic fibre-reinforced elastic media, we replace medium  $M_2$  by vacuum. Since the boundary  $z = 0$  is adjacent to vacuum, it is free from surface traction. So the stress boundary condition in this case may be expressed as

$$\tau_{31} = 0, \quad \tau_{33} = 0 \quad \text{on} \quad z = 0.$$

Thus Eqs (4.2d) and (4.2f) reduce to

$$\left( 2i\beta_1 + \gamma_1 + \beta_1^2\gamma_1 \right) A + \left( 2i\beta_2 + \gamma_2 + \beta_2^2\gamma_2 \right) B = 0, \tag{5.1a}$$



$$\begin{aligned}
 &A\left[\lambda\left(\beta_1^2 - I\right) + 2\mu_T\left(\beta_1^2 - i\beta_1\gamma_1\right) + \alpha(-I + i\beta_1\gamma_1)\right] + \\
 &+ B\left[\lambda\left(\beta_2^2 - I\right) + 2\mu_T\left(\beta_2^2 - i\beta_2\gamma_2\right) + \alpha(-I + i\beta_2\gamma_2)\right].
 \end{aligned}
 \tag{5.1b}$$

Eliminating  $A$  and  $B$  from Eqs (5.1a) and (5.1b), we have

$$\begin{aligned}
 &\left[2i\beta_1 + \gamma_1\left(\beta_1^2 + I\right)\right]\left[\lambda\left(\beta_2^2 - I\right) + 2\mu_T\left(\beta_2^2 - i\beta_2\gamma_2\right) + \alpha(-I + i\beta_2\gamma_2)\right] + \\
 &- \left[2i\beta_2 + \gamma_2\left(\beta_2^2 + I\right)\right]\left[\lambda\left(\beta_1^2 - I\right) + 2\mu_T\left(\beta_1^2 - i\beta_1\gamma_1\right) + \alpha(-I + i\beta_1\gamma_1)\right] = 0.
 \end{aligned}
 \tag{5.2}$$

Equation (5.2) is the wave velocity equation for Rayleigh waves in a fibre-reinforced solid elastic medium under the influence of gravity.

From Eq.(5.2), we see that Rayleigh waves depend on the gravity and the parameters for fibre-reinforcement of the material medium.

In the absence of gravity, this equation is in complete agreement with the corresponding classical result (Sengupta and Acharya, 1979).

Again in the absence of gravity and writing  $\mu_L = \mu_L - \mu_T + \mu_T$  and making  $\alpha, \beta$  and  $|\mu_L - \mu_T|$  all tend to zero, Eq.(5.2) reduces to the following form

$$\left(2 - \frac{\rho c^2}{\mu_T}\right)^2 = 4\left(I - \frac{\rho c^2}{\lambda + 2\mu_T}\right)^{1/2} \left(I - \frac{\rho c^2}{\mu_T}\right)^{1/2},
 \tag{5.3}$$

which gives the wave velocity equation for Rayleigh waves in an isotropic elastic medium.

**Love waves:** To investigate the possibility of Love waves in a fibre-reinforced elastic solid media, we replace medium  $M_2$  which is obtained by two horizontal plane surfaces at a distance  $H$ -apart, while  $M_1$  remains infinite.

For the medium  $M_1$ , the displacement component  $v$  remains same as in general case given by Eq.(3.4).

For the medium  $M_2$ , we preserve the full solution, since the displacement component along  $y$ -axis, i.e.,  $v$  no longer diminishes with increasing the distance from the boundary surface of the two media.

Thus, 
$$v' = E' \exp\{K'_3 z + i\omega(x - ct)\} + F' \exp\{-K'_3 z + i\omega(x - ct)\}.$$
 (5.4)

In this case, the boundary conditions are

- (i)  $v$  and  $\tau_{32}$  are continuous at  $z = 0$
- (ii)  $\tau'_{32} = 0$  at  $z = -H$ .

Applying boundary conditions (i) and (ii) and using Eqs (3.4), (4.1) and Eq.(5.4), we get

$$E = E' + F',$$
 (5.5)

$$-p_3(\mu_T)E = (\mu'_T) [K'_3 E' - K'_3 F'],$$
 (5.6)

$$K_3' E' e^{-K_3' H} - K_3' F' e^{K_3' H} = 0. \quad (5.7)$$

On eliminating the constant  $E$ ,  $E'$  and  $F'$  from Eqs (5.5), (5.6) and (5.7)

$$\tanh(K_3' H) = \frac{K_3(\mu_T)}{K_3'(\mu_T')}. \quad (5.8)$$

Thus Eq.(5.8) gives the wave velocity equation for Love waves in a fibre-reinforced elastic solid medium under the influence of gravity. Clearly, fibre-reinforcement plays a vital role in the propagation of Love waves whereas the presence of gravitational field cannot influence the same. Moreover, the thickness of the fibre-reinforced layer has a pronounced effect on the propagation of Love waves.

If gravity and parameters for fibre-reinforcement of material medium are not considered, Eq.(5.8) is in complete agreement with the corresponding classical result of Bullen.

## 6. Discussion and conclusions

In the light of the above analysis, the following conclusions can be made.

1. The present study reveals the effects of elasticity, gravity and fibre-reinforcement parameters on the wave velocity equations corresponding to Stoneley waves, Rayleigh waves and Love waves. The results of this analysis are useful in circumstances where these effects cannot be neglected.
2. Rayleigh wave velocity is modulated to a considerable extent by the presence of gravity and fibre-reinforcement parameters. Further, it is noted that due to the presence of gravity wave velocity Eq.(5.2) depends on the particular value of  $\omega$  which indicates the dispersion of the general wave form. It can also be noted that when  $\omega$  is large and as a consequence the wave-length is small, the effect of gravity is sufficiently small. Again if  $\omega$  is small and the wave-length is large, the effect of gravity has a significant effect on wave velocity. The results are in complete agreement with the corresponding classical results when fibre-reinforcement parameters and gravitational fields are neglected.
3. It is noted from Eq.(5.8) that gravity cannot influence Love wave propagation. Frequency ( $\omega$ ), thickness of the layer ( $H$ ) and fibre-reinforcement parameters have a salient influence on Love wave propagation. If gravity and parameters for fibre-reinforcement of the material medium are neglected, the dispersion equation is in complete agreement with the corresponding classical result.
4. It is noted that the wave velocity equation of Stoneley waves is very similar to the corresponding problem in the classical theory of elasticity. Here also waves are dispersed due to the presence of gravity, frequency and fibre-reinforcement parameters. Also, the wave velocity equation of this generalized type of surface waves under the influence of gravity in anisotropic fibre-reinforced elastic solid media is in complete agreement with the corresponding classical results when gravity and parameters for fibre-reinforcement of the material medium, are neglected.

Further, the solution of the wave velocity equation for Stoneley waves cannot be determined by easy analytical methods. However, we can apply numerical techniques to solve this equation by choosing suitable values of physical constants for both media  $M_1$  and  $M_2$ .

## Nomenclature

- $e_{ij}$   $i,j = 1,2,3$ , – strain components  
 $g$  – acceleration due to gravity  
 $\alpha, \beta, (\mu_L - \mu_T)$  – reinforced anisotropic elastic parameters  
 $\lambda, \mu_T$  – elastic parameters

$\phi, \psi$  and  $v$  – associated with P waves. SV waves and SH waves

$\rho$  – density of the material medium

$\tau_{ij}, j=1, 2, 3$  – stress components

$\nabla^2 \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2}$  – Laplacian operator

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