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DIFFERENT SYNCHRONIZATION SCHEMES FOR CHAOTIC RIKITAKE SYSTEMS

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This paper presents the chaos synchronization by designing a different type of controllers. Firstly, we propose the synchronization of bi-directional coupled chaotic Rikitake systems via hybrid feedback control. Secondly, we study the synchronization of unidirectionally coupled Rikitake systems using hybrid feedback control. Lastly, we investigate the synchronization of unidirectionally coupled Rikitake chaotic systems using tracking control. Comparing all the results, finally, we conclude that tracking control is more effective than feedback control. Simulation results are presented to show the efficiency of synchronization schemes.

Key words: chaotic system, chaos synchronization, chaos control, Rikitake system.

1. Introduction

Since the pioneer works by Ott *et al*. (1990) and Pecorra and Carroll (1990), chaos control and synchronization have received increasing attention due to theoretical challenges and potential applications to various disciplines. Synchronization in biological systems is one of the fascinating areas that has attracted a lot of attention.

The analysis of synchronization phenomena in the evaluation of dynamical system has been the subject of active investigation. Chaos synchronization has become very important in the non-linear science over the last two decades, due to its potential applications in many areas such as secure communication, information processing, biological systems, chemical reactions, neural networks and in engineering. Usually, two dynamical systems are called synchronized if the distance between their corresponding states converges to zero as time goes to infinity.

This type of synchronization is known as identical synchronization (Pecorra and Carroll, 1990). Using linear and non linear feed back control chaos synchronization has been presented in various chaotic systems. Synchronization of unified chaotic systems using adaptive feedback control was studied by Lu and Chen in 2002. Park (2005) studied controlling chaotic systems via nonlinear feedback control. Chen *et al*. (2006) proposed generating hyperchaotic Lu attractor via state feedback control. Synchronization between two different noise perturbed chaotic systems with fully unknown parameters was proposed by Sun and Cao (2007). Poria *et al*. (2007) investigated adaptive synchronization of two coupled chaotic neuronal systems. Recently Khan *et al*. (2011) investigated control strategies for unified chaotic systems using different type of control.

In this paper firstly, we discuss the synchronization between two bidirectionally coupled chaotic Rikitake systems via hybrid feedback control.

Secondly, we study the synchronization between two identical Rikitake systems using hybrid feedback control and lastly, we investigate the synchronization of two identical Rikitake chaotic systems using tracking control.

2. Design of hybrid controller of bidirectionally couple chaotic systems

A dynamical system can be written as

$$\dot{x} = f(x). \tag{2.1}$$

This system can also be expressed as

$$\dot{x} = Ax + B\Psi(x) \tag{2.2}$$

where $A \in R^{n\times n}$, $B \in R^{n\times m}$ are constant matrices, $\Psi(x): R^n \to R^n$ is a non-linear vector function. We consider the following type of bidirectionally coupled chaotic systems

$$\dot{x} = Ax + B\left[\Psi(x) + u_1\right], \tag{2.3}$$

$$\dot{y} = Ay + B[\Psi(y) + u_2] \tag{2.4}$$

where $x \in R^n$, $y \in R^n$, $A \in R^{n\times n}$, $B \in R^{n\times m}$, $\Psi : R^n \to R^n$ is non-linear vector functions. The synchronization errors between systems (2.3) and (2.4) are defined as $e = \left(e_1, e_2, ..., e_n\right)^T = \left(x_1 - y_1, x_2 - y_2, ..., x_n - y_n\right)^T$. Then the error dynamical system is

$$\dot{e} = Ae + B\left[\Psi(x) - \Psi(y) + u_1 - u_2\right]. \tag{2.5}$$

In order to make system (2.3) and (2.4) synchronizable, the coupling functions u_1 and u_2 should be properly chosen.

Let $u_1 = u_{11} + u_{12}$ and $u_2 = u_{21} + u_{22}$, where $u_{11} = \Psi(y)$, $u_{12} = -Ky$ and $u_{21} = \Psi(x)$, $u_{22} = -Kx$ and $K = \begin{pmatrix} k_1 & 0 & 0 \\ 0 & k_2 & 0 \\ 0 & 0 & k_3 \end{pmatrix}$, denote the feedback matrix. Obviously u_{11}, u_{21} are non-linear controllers and u_{12}, u_{22} are linear controller, so u_1 and u_2 are hybrid controllers.

Theorem: If the matrix $A + BK$ has all eigen values with negative real parts, then the origin will be an asymptotically stable fixed point of system (2.5).

Proof: Choosing the controller u_1 and u_2 properly, the error system (2.5) can be written as

$$\dot{e} = (A + BK)e. \tag{2.6}$$

Now by the theory of linear dynamical system if the matrix $A + BK$ has all eigen values with negative real parts, then the origin of the error system will be a globally asymptotically stable fixed point. Therefore in this case the bi-directionally coupled systems (2.3) and (2.4) will synchronize.

Example. We shall now discuss the efficiency of our scheme taking coupled chaotic Rikitake systems. The Rikitake system can be described by the following system of differential equations.

$$\dot{x}_1 = x_2 x_3 - a_2 x_1 ,$$
$$\dot{x}_2 = (x_3 - b_2)x_1 - a_2 x_2 , \tag{2.7}$$
$$\dot{x}_3 = 1 - x_1 x_2$$

where $x_1, x_2, x_3 \in R^n$ are state variables and a_2, b_2 are real constants. System (2.7) is found to be chaotic when $a_2 = 2$ and $b_2 = 5$.

According to our choice of controller the coupled systems are

$$\dot{x}_1 = x_2 x_3 - a_2 x_1 + y_2 y_3 - k_1 y_1 ,$$
$$\dot{x}_2 = (x_3 - b_2)x_1 - a_2 x_2 + y_1 y_3 - k_2 y_2 , \tag{2.8}$$
$$\dot{x}_3 = 2 - x_1 x_2 - y_1 y_2 + y_3 - k_3 y_3 ,$$

and

$$\dot{y}_1 = y_2 y_3 - a_2 y_1 + x_2 x_3 - k_1 x_1 ,$$
$$\dot{y}_2 = (y_3 - b_2)y_1 - a_2 y_2 + x_1 x_3 - k_2 x_2 , \tag{2.9}$$
$$\dot{y}_3 = 2 - y_1 y_2 - x_1 x_2 + x_3 - k_3 x_3 .$$

Therefore the error system is

$$\dot{e}_1 = -(a_2 - k_1)e_1 ,$$
$$\dot{e}_2 = -b_2 e_1 + (k_2 - a_2)e_2 , \tag{2.10}$$
$$\dot{e}_3 = (k_3 - 1)e_3 .$$

Now for a suitable choice of *K* the matrix $A + BK$ has all eigen values with negative real parts, and then the drive system synchronizes with the response system.

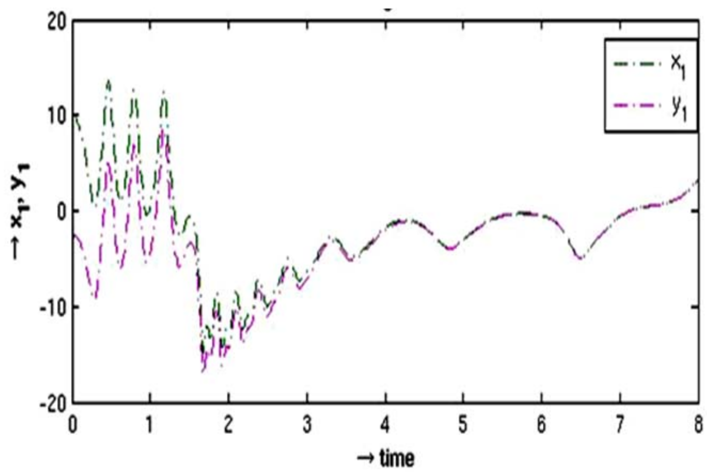

Fig.1.

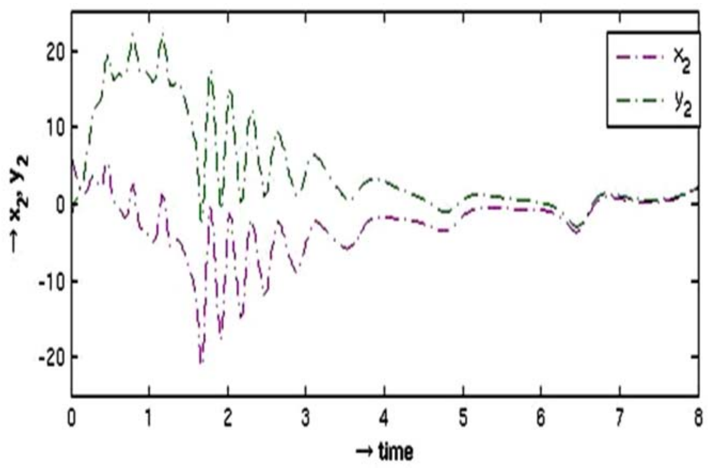

Fig.2.

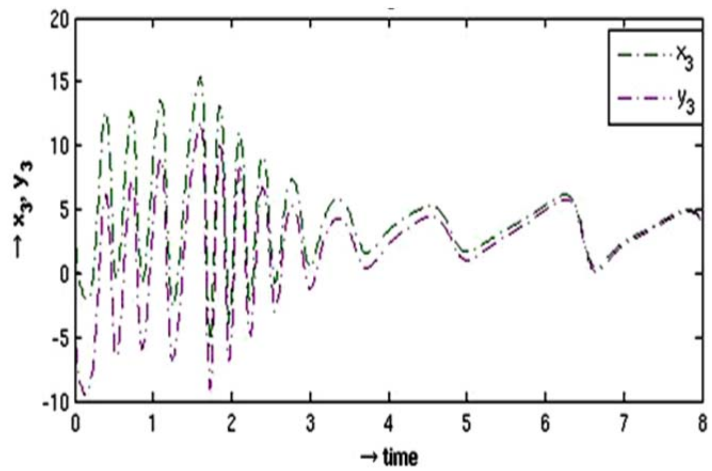

Fig.3.

Fig.1-Fig.3. Represents the trajectories of (x_1, y_1), (x_2, y_2) and (x_3, y_3).

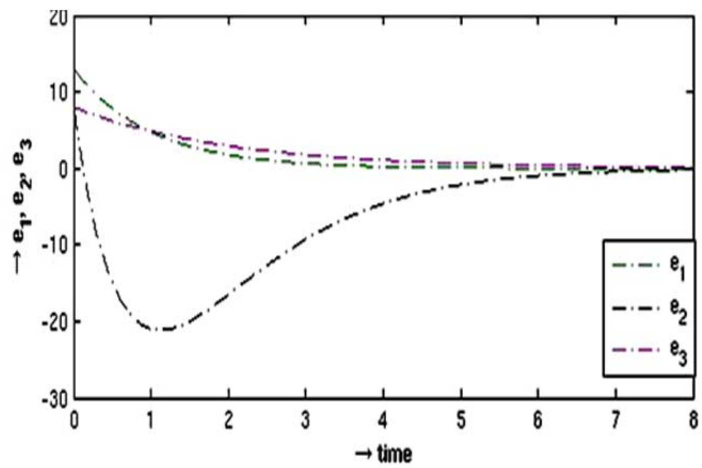

Fig.4. Shows time evolution of the synchronization errors.

We choose $A = \begin{pmatrix} -2 & 0 & 0 \\ -5 & -2 & 0 \\ 0 & 0 & -1 \end{pmatrix}$, $B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, and $K = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & .5 \end{pmatrix}$.

For numerical simulation, the fourth order Runge-Kutta method is used. We select the parameters $(a_2, b_2) = (2, 5)$. The initial conditions of system (2.8) and (2.9) are chosen as $(x_1(0), x_2(0), x_3(0)) = (10, 6, 3)$ and $(y_1(0), y_2(0), y_3(0)) = (-3, -2, -5)$, so the initial synchronization errors are $(e_1(0), e_2(0), e_3(0)) = (13, 8, 8)$. The trajectories of the x_1, state of the drive system and y_1, state of the response system are shown in Fig.1. The trajectories of x_2 and y_2 are shown in Fig.2. and the trajectories of x_3 and y_3 are shown in Fig.3. The figures confirm the synchronization between the two chaotic bidirectionally systems. Time evolution of the synchronization errors goes to zero as shown in Fig.4.

3. Synchronization of coupled Rikitake systems via hybrid feedback control

In this section, we discuss the synchronization of the coupled Rikitake systems via hybrid feedback control. The Rikitake system (2.7) can be rewritten as

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{pmatrix} = \begin{pmatrix} -a_2 & 0 & 0 \\ -b_2 & -a_2 & 0 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} + \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_2 x_3 \\ x_1 x_3 \\ 1 + x_3 - x_1 x_2 \end{pmatrix}. \tag{3.1}$$

Comparing Eq.(3.1) with Eq.(2.2), we get

$$A = \begin{pmatrix} -a_2 & 0 & 0 \\ -b_2 & -a_2 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \Psi(x) = \begin{pmatrix} x_2 x_3 \\ x_1 x_3 \\ 1 + x_3 - x_1 x_2 \end{pmatrix}.$$

Now, $A - BK = \begin{pmatrix} -2 - k_1 & 0 & 0 \\ -5 & -2 - k_2 & 0 \\ 0 & 0 & -1 - k_3 \end{pmatrix}$ where $a_2 = 2$ and $b_2 = 5$.

The characteristic equation of $A - BK$ is

$$\lambda^3 + \lambda^2 (k_1 + k_2 + k_3 + 5) + \lambda (k_1 k_2 + k_1 k_3 + k_2 k_3 + 3k_1 + 3k_2 + 4k_3 + 8) + \\ + (k_1 + 2)(k_2 + 2)(k_3 + 1) = 0.$$

According to Routh-Hurwitz, the matrix $A - BK$ is negative definite if

$$\begin{aligned} & k_1 + k_2 + k_3 + 5 > 0, \\ & k_1 k_2 + k_1 k_3 + k_2 k_3 + 3k_1 + 3k_2 + 4k_3 + 8 > 0, \\ & (k_1 + 2)(k_2 + 2)(k_3 + 1) > 0, \\ & (k_1 + k_2 + k_3 + 5)(k_1 k_2 + k_1 k_3 + k_2 k_3 + 3k_1 + 3k_2 + 4k_3 + 8) > (k_1 + 2)(k_2 + 2)(k_3 + 1), \end{aligned} \tag{3.2}$$

are satisfied when $k_1 = 1, k_2 = 1$ and $k_3 = .5$ then $A - BK$ is negative definite. Therefore the controller u_1 and u_2 can be chosen as

$$u_1 = \begin{pmatrix} x_2x_3 - y_2y_3 \\ x_1x_3 - y_1y_3 \\ -x_1x_2 + y_1y_2 + x_3 - y_3 \end{pmatrix} \text{ and } u_2 = K(x-y) = \begin{pmatrix} k_1 & 0 & 0 \\ 0 & k_2 & 0 \\ 0 & 0 & k_3 \end{pmatrix}\begin{pmatrix} x_1 - y_1 \\ x_2 - y_2 \\ x_3 - y_3 \end{pmatrix}.$$

Therefore the response system becomes

$$\begin{aligned} \dot{y}_1 &= -(a_2+1)y_1 + x_1 + x_2x_3, \\ \dot{y}_2 &= -b_2y_1 - (a_2+1)y_2 + x_2 + x_1x_3, \\ \dot{y}_3 &= \frac{1}{2}(x_3 - y_3) - x_1x_2 + 1. \end{aligned} \tag{3.3}$$

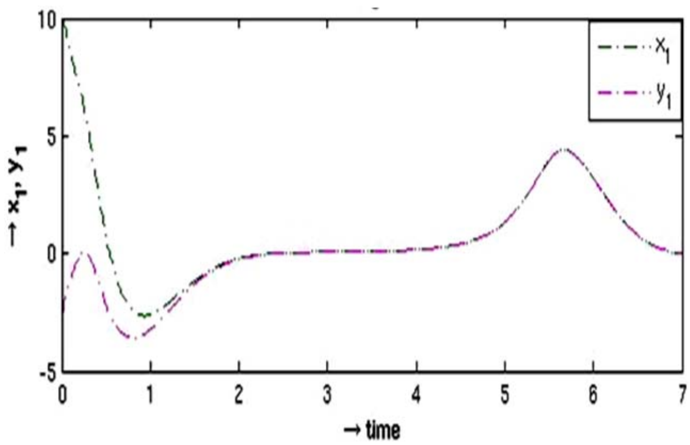

Fig.5.

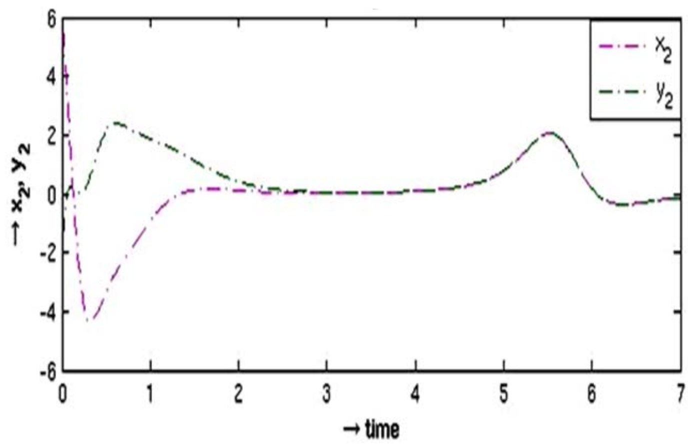

Fig.6.

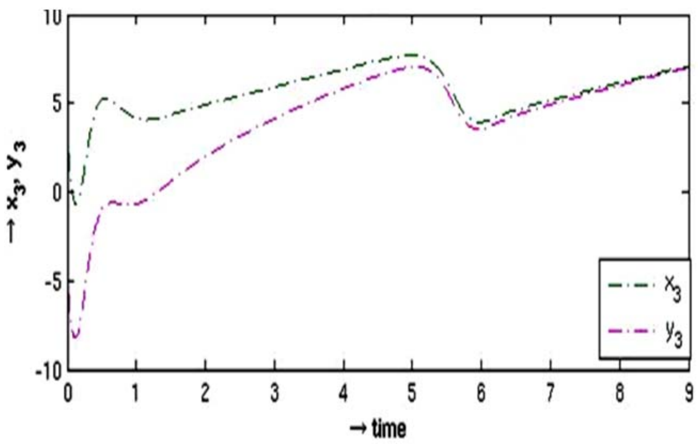

Fig.7.

Fig.5-Fig.7. Present the trajectories of (x_1, y_1), (x_2, y_2) and (x_3, y_3).

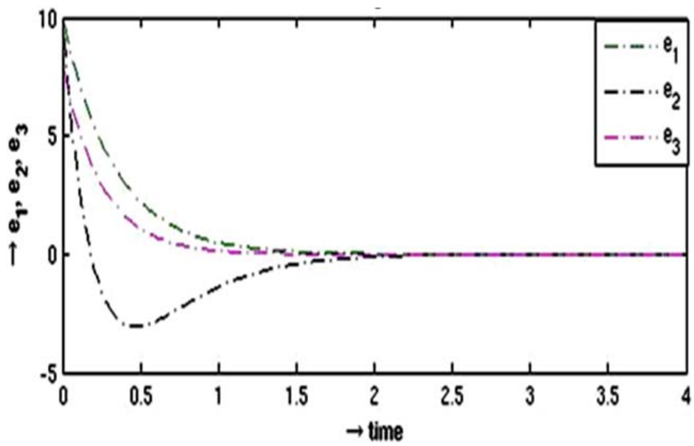

Fig.8. Shows time evolution of the synchronization errors.

For numerical simulation, the fourth order Runge-Kutta method is used. We select the parameter $(a_2, b_2) = (2, 5)$. The initial conditions of system (3.1) and (3.3) are chosen as $(x_1(0), x_2(0), x_3(0)) = (10, 6, 3)$ and $(y_1(0), y_2(0), y_3(0)) = (-3, -2, -5)$, so the initial values of errors are (13,8,8). The trajectories of x_1, state of the drive system and y_1, state of the response system are shown in Fig.5. The trajectories of x_2 and y_2 are shown in Fig.6 and the trajectories of x_3 and y_3 are shown in Fig.7. The figures confirm synchronization of the coupled systems. Time evolution of the synchronization errors converges to zero as shown in Fig.8.

4. Design of tracking controller

For discussing the synchronization of unidirectional coupled chaotic system via tracking control we assume that system (2.2) is the drive system. Upon introduction of the control variable $U \in R^n$, the controlled response system is given by

$$\dot{y} = Ay + B\Psi(y) + U \tag{4.1}$$

where $y \in R^n$ denotes the state vector of the response system. The main problem is to design a controller U which synchronizes the state of both drive and response systems. We substract (2.2) from Eq.(4.1) and we get

$$\dot{e} = Ae + B(\Psi(y) - \Psi(x)) + U \tag{4.2}$$

where $e = y - x$. The aim is to make $\lim_{t\to\infty} \|e(t)\| = 0$. Let the Lyapunov error function be $V(e) = \frac{1}{2}e^T e$, where $V(e)$ is a positive definite function. Assume that the parameters of the drive and response systems are known and the states of both systems are measurable. We may achieve the synchronization by selecting the controller U to make the first derivative $V(e)$, i.e., $\dot{V}(e) < 0$. Then the state of the response and drive system is synchronized asymptotically.

Example. Synchronization of Rikitake system via tracking control

Let the dynamical system (2.7) be the drive system, then the controlled response Rikitake system is given by the following

$$\begin{aligned} \dot{y}_1 &= y_2 y_3 - a_2 y_1 + u_1, \\ \dot{y}_2 &= (y_3 - b_2) y_1 - a_2 y_2 + u_2, \\ \dot{y}_3 &= 1 - y_1 y_2 + u_3. \end{aligned} \tag{4.3}$$

Let us define the error between the trajectories of the response and drive Rikitake system as $e_i = y_i - x_i$ $(i = 1,2,3)$. Therefore the error system is

$$\begin{aligned} \dot{e}_1 &= -a_2 e_1 + y_2 y_3 - x_2 x_3 + u_1, \\ \dot{e}_2 &= -b_2 e_1 - x_1 x_3 + y_1 y_3 - a_2 e_2 + u_2, \\ \dot{e}_3 &= x_1 x_2 - y_1 y_2 + u_3. \end{aligned} \tag{4.4}$$

If we choose the controller as

$$\begin{aligned} u_1 &= x_2 x_3 - y_2 y_3, \\ u_2 &= x_1 x_3 - y_1 y_3 + b_2 e_1, \\ u_3 &= y_1 y_2 - x_1 x_2 - e_3, \end{aligned} \tag{4.5}$$

then the coupled Rikitake systems will get synchronized.

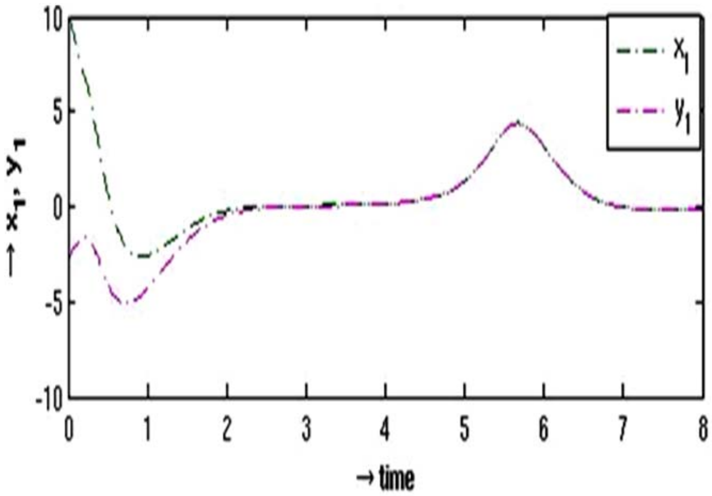

Fig.9.

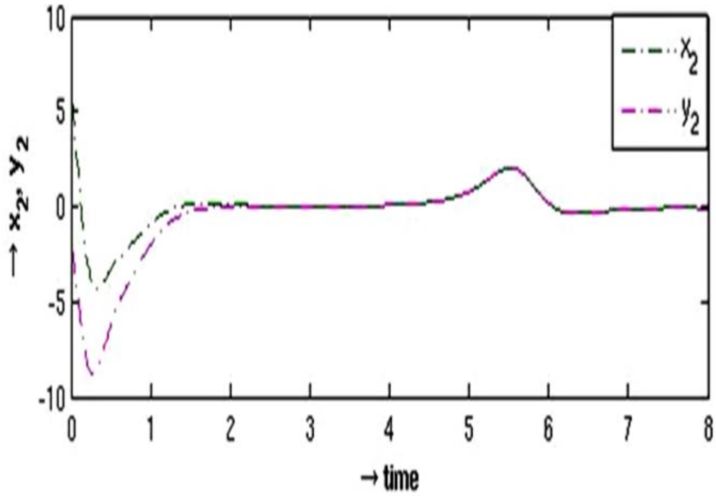

Fig.10.

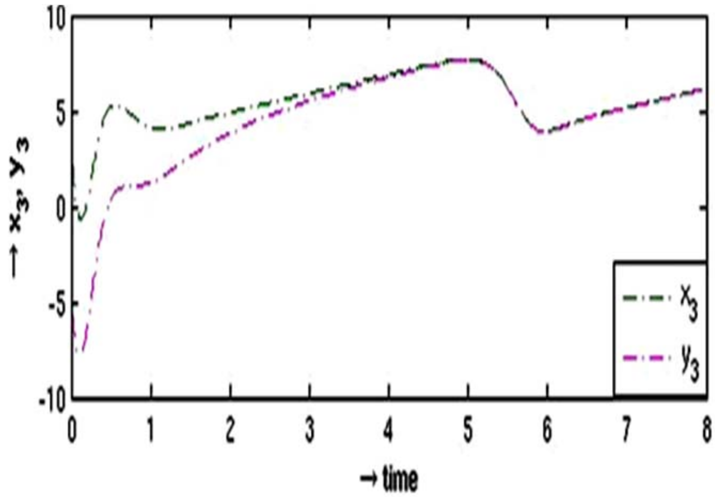

Fig.11.

Fig.9-Fig.11. Present the trajectories of (x_1, y_1), (x_2, y_2) and (x_3, y_3).

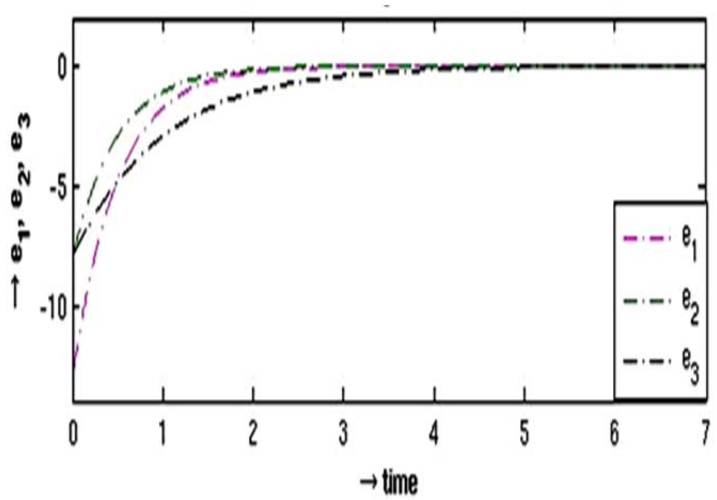

Fig.12. Shows time evolution of the synchronization errors.

The parameters of Rikitake systems are $(a_2,b_2)=(2,5)$. The initial conditions for the driving system and driven system are given respectively by $(x_1(0),x_2(0),x_3(0))=(10,6,3)$ and $(y_1(0),y_2(0),y_3(0))=(-3,-2,-5)$, so the initial values of error system are $(e_1(0),e_2(0),e_3(0))=(13,8,8)$. The trajectories of x_1, state of the drive system and y_1, state of the response system are shown in Fig.9. The trajectories of x_2 and y_2 are shown in Fig.10. and the trajectories of x_3 and y_3 are shown in Fig.11. The figures confirm the synchronization of the systems. Time evolution of the synchronization errors goes to zero as shown in Fig.12.

6. Conclusions

This paper investigates the synchronization by designing a different type of controllers, which include bi-directional coupled chaotic systems using hybrid feedback control and unidirectional coupled chaotic systems using hybrid feedback control and tracking control. We apply the above three controller methods to Rikitake systems to prove the feasibility and effectiveness of the proposed scheme. Comparing all the results finally we conclude from numerical simulation results that tracking control is more effective than feedback control.

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Nomenclature

$A \in R^{n\times n}, B \in R^{n\times n}$ – constant matrix
a_2, b_2 – real constants
K – feedback matrix
u_{11}, u_{21} – non-linear controller
u_{12}, u_{22} – linear controller
$V(e)$ – Lyapunov error function

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