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RESIDUAL ERROR BASED ADAPTIVE METHOD WITH AN OPTIMAL VARIABLE SCALING PARAMETER FOR RBF INTERPOLATION

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In infinitely smooth Radial Basis Function (RBF) based interpolation, the scaling parameter plays an important role to obtain an accurate and stable numerical solution. When this method is applied to interpolate a function with sharp gradients, then adaptive methods will also play a significant role in determining an optimal number of centers according to the user desired accuracy. In this article, we test an optimization algorithm developed using the nonlinear optimization to find a scaling parameter for RBF along with an adaptive residual subsampling method [1] RBF interpolation. In this process, at each stage of adoption, the available optimal shape parameters have been obtained by solving the system of non-linear equations.

Key words: radial basis function, adaptive residual subsampling, optimal shape parameter.

1. Introduction

Grid free Radial Basis Function (RBF) approximation methods are used to solve the problems related to engineering and scientific computing. RBF approximation schemes are helpful due to their adaptability and efficiency of handling the scattered data in complex domains. RBF methods are utilized basically for multivariable scattered data approximation [2,3]. The hypothetical foundations of RBF methods are presented in [4, 5]. Some of the disadvantages of the RBF approximation methods are, ill conditioning of the global matrix with the increment of number of collocation points and so the condition number of the matrix. One more issue with RBF approximations is the trouble in assuring a proper value of the scaling parameter or shape parameter [6-9].

In recent years, radial basis function approximation schemes have become popular in solving the initial and boundary value problems for partial differential equations (PDEs). One of the most widely used RBF approximation method to solve these kinds of problems is the Kansa method [10]. Numerical solutions of the PDEs using the Kansa RBF method may be found in [11]. In RBF approximation methods, the RBF contains a free scaling parameter which plays a very important role in ensuring the accuracy of the proposed method. In the majority of the research papers the analysists have chosen the scaling parameter by trial and error or some other ad-hoc means [12, 13].

Several methods for selecting the scaling parameter for the RBF interpolants were suggested by Hardy [14], Franke [4]. Carlson and Foley [13] studied dependence of the accuracy of the shape parameter for RBFs like multi quadric (MQ) and inverse multiquadric (IMQ) interpolants. Foley [15] studied the near optimal shape

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parameter selection for MQ interpolant. Kansa and Carlson [12] discussed the accuracy of MQ using variable shape parameter. Several numerical techniques exist for selecting the value of the scaling parameter optimally by minimizing an error function. A well-known method in statistics was proposed by Rippa [9] based on the LOOCV (leave one out cross validation) approximation method. By minimizing the cost function, the optimal value of the shape parameter is determined by the LOOCV approximation method. Fasshaur and Zhang [7], Schuerer [16], Roque and Ferreira [17], are some authors who worked on scaling parameter optimization by an error function minimization. Sanyasiraju and Satyanarayana [18] proposed a meshless local scheme for optimization of the scaling parameter. Larsson and Fornberg [19], Cheng [6], Scheuerer [16], Fornberg *et al.* [20], Luh [21], dealt with the impact of the scaling parameter in RBF approximations. Cavoretto *et al.* [22] discussed the determination of the optimal scaling parameter in RBF-PU interpolation.

Jafar Biazar and Mohammad Hosami [23] presented a procedure to find an interval, instead of the value of the shape parameter without minimizing and estimating an error function. Jankowska et al [24] discussed the two-dimensional nonlinear boundary value problems (BVPs) by including the scaling parameter value in the set of unknowns. Solution of these nonlinear equations gives the optimal shape parameter value. These authors have extended their work to solve 2^{nd} and 4^{th} order nonlinear BVPs [25].

Over the last few years, some of the authors have tested RBF methods with distinct adaptive schemes for time-dependent and time-independent problems. Hon [26] proposed an adaptive method using multi-quadrics for a boundary layer problem in which more points are allocated to the boundary layer. Schaback [27] and Hon [28] suggested an adaptive method based on greedy algorithm with the help of compactly supported RBFs for interpolation and collocation problems. Bozzini [29] applied B-spline techniques with radial basis functions to scaled multi quadrics for interpolation. Behrens [30] combined an adaptive semi-Lagrangian technique by using local thin plate spline interpolation. In [31] the authors extended their work for nonlinear transport equation. Sarra [32] used an adaptive method for RBFs to solve time dependent PDEs in which an algorithm of simple moving grid was used in low-order finite difference methods.

A new method called adaptive residual subsampling was proposed by Driscroll and Heryudono in [1] which is applicable to interpolation, BVPs and initial BVPs. In this method centres can be either added or eliminated in view of the residuals calculated at a finer point set and the shape parameter of RBFs was adopted based on node spacing. Covoretto and Rossi [33] presented an adaptive technique based on residual subsampling to solve elliptical PDEs by using the method of partition of unity. In [34] they proposed an adaptive LOOCV algorithm to solve elliptical PDEs. In [35] the authors extended their work using a two-stage adaptive scheme and in [36] un-symmetric and symmetric collocation problems were solved using an adaptive scheme.

The principal motivation of this article is to apply an adaptive residual subsampling method [1] along with an optimization algorithm [24, 25] for interpolation of functions in one dimension with sharp gradients. When we apply this method to interpolate a function, the adaptive algorithm will play a crucial role in determining an optimal number of centers according to the desired accuracy specified by the user. The variable optimal shape parameters have been found by solving the nonlinear system which is obtained through the interpolation problem. Since the adaption is taking place at each stage, the optimization algorithm and adaptive residual subsampling method [1] together will produce an optimal number of centers as well as an optimal value of the variable scaling parameter.

The paper is organized as follows. Section 2 presents RBF based interpolation. Section 3 deals with the strategies to select an optimal value of the shape parameter. Section 4 discusses the adaptive residual subsampling method. Section 5 describes the algorithm of the proposed method. Numerical examples are discussed in Section 6.

2. Radial basis function interpolation

Define the interpolating function $f:\mathbf{\Omega}\to\mathbb{R}$, where $\mathbf{\Omega}\subset\mathbb{R}^d$, from a set of sample values $\left\{f\left(\vec{x}_i\right)\right\}_{i=1}^N$ on a discrete set $X=\left\{\vec{x}_i\right\}_{i=1}^N\subset\mathbf{\Omega}$. f is approximated by $\mathcal{U}\left(\vec{x};\epsilon\right)$, where

$$\mathcal{U}(\vec{x};\epsilon)=\sum_{j=1}^{N}\alpha_j \varnothing_\epsilon\left(\vec{x}-\vec{x}_j\right) \tag{2.1}$$

here ϵ is the shape parameter. The interpolation conditions are:

$$\mathcal{U}(\vec{x}_i)=f(\vec{x}_i),\ i=1,2,\ldots,N\ . \tag{2.2}$$

On applying conditions (2.2) in Eq.(2.1), we get the following linear system of equations,

$$A\boldsymbol{\alpha}=\boldsymbol{f} \tag{2.3}$$

where $A_{ij}=\varnothing_\epsilon\left(\vec{x}_i-\vec{x}_j\right),\ i,j=1,\ldots..,N, \boldsymbol{\alpha}=\left(\alpha_1,\ldots.,\alpha_N\right)^T$ and $\boldsymbol{f}=\left(f_1,\ldots.,f_N\right)^T$. The coefficients $\{\alpha_i\}_{i=1}^N$ in Eq.2.1 are determined by solving system (2.3).
Common choices of $\varnothing_\epsilon(r)$ are:

- Gaussian $(GA), \varnothing_\epsilon(r)=e^{-\epsilon^2 r^2}$.
- Multi Quadric (MQ), $\varnothing_\epsilon(r)=\left(1+\epsilon^2 r^2\right)^{1/2}$.
- Inverse Multi Quadric (IMQ), $\varnothing_\epsilon(r)=\left(1+\epsilon^2 r^2\right)^{-1/2}$.
- Wendland $(W2)$, $\varnothing_\epsilon(r)=(1-\epsilon r)^4(4\epsilon r+1)$, where $\epsilon>0$ is a shape parameter.
- Thin plate spline, $\varnothing(r)=r^2\ln r$.

The matrix A in system (2.3) is nonsingular for Multi Quadric (MQ), Inverse Multi Quadric (IMQ) and Gaussian. This is also valid for thin plate spline by certain changes on the interpolant by the addition of a polynomial term [37, 38]. The infinitely smooth RBFs like multi quadric, inverse multi quadric and Gaussian give spectral rate of convergence, piecewise smooth RBFs gives an algebraic rate of convergence [37, 38]. In the present work, the numerical results are obtained with MQ-RBF.

3. Optimization of shape parameter

We have certain numerical schemes for choosing the optimal scaling parameter value by the minimization of an error function. Rippa [9], Fasshaur and Zhang [7], Schuerer [16], Roque and Ferreira [17] are among the few who have worked on the optimization of the scaling parameter using this method. Some authors have examined the effect of the scaling parameter on the precision of the approximation through ad-hoc criteria. Rippa [9] was the person who initialized an algorithm based on the LOOCV approximation method which is a well-known technique in statistics. Mathematicians like Hardy [14], Franke [4], Foley and Carlson [13], Carlson and Kansa [12], Foley [15], Wright and Fornberg [39], Larsson and Fornberg [19], Fasshauer and Zhang [7], Scheuerer [16], Cheng [4], Fornberg *et al.* [28], Luh [32] have worked on the importance of the scaling parameter in RBF approximation methods.

Jafar Biazar and Mohammad Hosami [23] presented a method to find an interval, in place of the scaling parameter value without minimization to estimate the error function. Their method depends on addition of a loop on the scaling parameter without minimizing the error norm. They have successfully determined an interval for the appropriate values of the shape parameter by considering the numerical convergence behaviour of the given problem. Therefore R_ϵ is the interval of the shape parameter. From Eq.(2.1) the function $f(\vec{x})$ is approximated by $\mathcal{U}(\vec{x};\epsilon)$, where $\mathcal{U}(\vec{x};\epsilon)$ denotes the approximant $\mathcal{U}$ depending on ϵ. In this method, the aim is to obtain the desired accuracy δ, i.e.,

$$\mathcal{U} - f < \delta \,. \tag{3.1}$$

For a fixed number of nodes, the accuracy δ may not be achieved. One can determine a desired criterion according to required accuracy as:

$$\left|\mathcal{U}\left(\vec{x};\epsilon\right) - f\left(\vec{x}\right)\right| < \gamma, \forall \vec{x} \in \boldsymbol{\Omega} \tag{3.2}$$

where γ is a reachable accuracy. Choose the safer interval R_ϵ as the set values of the scaling parameter which satisfies (2.1), i.e.,

$$R_\epsilon = \{\epsilon : \left|\mathcal{U}\left(\vec{x};\epsilon\right) - f\left(x\right)\right| < \gamma, \forall \vec{x} \in \boldsymbol{\Omega}\} \,. \tag{3.3}$$

Taking $\vec{x}^* \in \boldsymbol{\Omega}$ and *n* fixed, we let:

$$R_{\epsilon,\vec{x}^*} = \{\epsilon : \left|\mathcal{U}\left(\vec{x}^*;\epsilon\right) - f\left(\vec{x}^*\right)\right| < \gamma\} \,. \tag{3.4}$$

Hence, to every value of $\epsilon \in R_{\epsilon,\vec{x}^*}$, the correlating values of $\mathcal{U}\left(\vec{x}^*;\epsilon\right)$ are equal to $f\left(\vec{x}^*\right)$ with the tolerance γ. In order to approximate $\mathcal{U}\left(\vec{x}^*\right)$, $R_{\epsilon,\vec{x}^*}$ contains better values for the scaling parameter. The suitable interval for R_ϵ is:

$$R_\epsilon = \bigcap\nolimits_{\vec{x}^* \in \boldsymbol{\Omega}} R_{\epsilon,\vec{x}^*} \,. \tag{3.5}$$

In practical approach we can select a few $\vec{x}^*$, in order to decide the set R_ϵ. Any value within the chosen interval is appropriate for the optimal value of the scaling parameter.

Jankowska *et al.* ([25] and [24]) discussed the Kansa-RBF collocation and RBF interpolation problems by including the scaling parameter in the set of unknown values. The solution of these nonlinear equations gives an optimal value of the variable scaling parameter. The present study associates a node dependent shape parameter to each RBF in the solution approximation [25]. The initial distribution of the auxiliary shape parameter is very important for the accuracy of the present scheme. Since the shape parameter in Eq.(2.2) appears nonlinearly, Eq.(2.2) is written as:

$$U\left(\boldsymbol{\alpha},\epsilon\right) = \left(\mathcal{U}_1,\ldots,\mathcal{U}_N\right)^T = 0 \,. \tag{3.6}$$

In particular, if $N = 3$, the corresponding nonlinear equations will be:

$$\mathcal{U}_1 = \alpha_1\sqrt{1+\epsilon_1^{\,2}\left(x_1 - x_1\right)^2} + \alpha_2\sqrt{1+\epsilon_2^{\,2}\left(x_1 - x_2\right)^2} + \alpha_3\sqrt{1+\epsilon_3^{\,2}\left(x_1 - x_3\right)^2} = 0 \,,$$

$$\mathcal{U}_2 = \alpha_1\sqrt{1+\epsilon_1^{\,2}\left(x_2 - x_1\right)^2} + \alpha_2\sqrt{1+\epsilon_2^{\,2}\left(x_2 - x_2\right)^2} + \alpha_3\sqrt{1+\epsilon_3^{\,2}\left(x_2 - x_3\right)^2} = 0 \,,$$

$$\mathcal{U}_3 = \alpha_1\sqrt{1+\epsilon_1^{\,2}\left(x_3 - x_1\right)^2} + \alpha_2\sqrt{1+\epsilon_2^{\,2}\left(x_3 - x_2\right)^2} + \alpha_3\sqrt{1+\epsilon_3^{\,2}\left(x_3 - x_3\right)^2} = 0$$

where the coefficients $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_N)^T$ and $\epsilon = (\epsilon_1, \ldots, \epsilon_N)^T$ have been obtained by solving the system of nonlinear Eqs (3.6).

4. Adaptive residual subsampling method for one dimensional interpolation

Since RBF approximation methods are meshless, they give flexibility to use adaptive methods for finding an optimal point set. In science and engineering, problems with rapid deformations occur in the given set of domains, such as sharp gradients, corners and topological changes. These changes are because of the non-linearity of the existing problem. In these circumstances adaptive procedures are preferred over the fixed grid methods. To achieve a desired accuracy and stability of the given problem adaptive methods [1] choose an optimal center by simultaneously adding/deleting the points based on the residuals.

In order to implement the adaptive residual subsampling method [1] to one dimensional interpolation problems, firstly the domain is discretized with uniformly spaced centers, and then the RBF interpolant of the given function is calculated. "The residual error at the centers midway between the connecting test points nodes are calculated. Nodes at which the residual error crosses a threshold θ_r will become centers and centers where error is under a threshold θ_c are terminated" [1]. This process is repeated until the approximation satisfies the desired thresholds.

5. Implementation of the proposed method

In our proposed method, we apply the adaptive residual subsampling method [1] combined with the optimization algorithm to RBFs for the approximation of functions in one dimension. In this method, at every stage of adoption, we obtain a desired number of centers and an optimal variable shape parameter. A pseudo-code of our proposed method is illustrated in the algorithm below.

Algorithm: ARSM with optimization algorithm

STEP 1 Fix the threshold θ_r and θ_c,
STEP 2 Initialize the centers X,
STEP 3 Initialize the interval for shape parameter ϵ using Eq.(3.5),
STEP 4 Initialize the shape parameter vector,
STEP 5 Compute $\boldsymbol{\alpha}$ and variable shape parameters by solving system (3.6),
STEP 6 Define a set of test points halfway between the nodes, i.e., Y,
STEP 7 Evaluate the residual error at these test points Y,
STEP 8 The points where the residual error $> \theta_r$, are added to X and the scaling parameter also determined at these points,
STEP 9 The points where the residual error $< \theta_r$, are removed and the scaling parameter is also removed at these points,
STEP 10 Repeat the steps 4-10 till the numerical solution satisfies the prescribed thresholds.

6. Numerical examples

To exhibit the efficiency of the proposed method, three numerical examples have been considered from [18] and compared with the results of RBF interpolation proposed in [18].

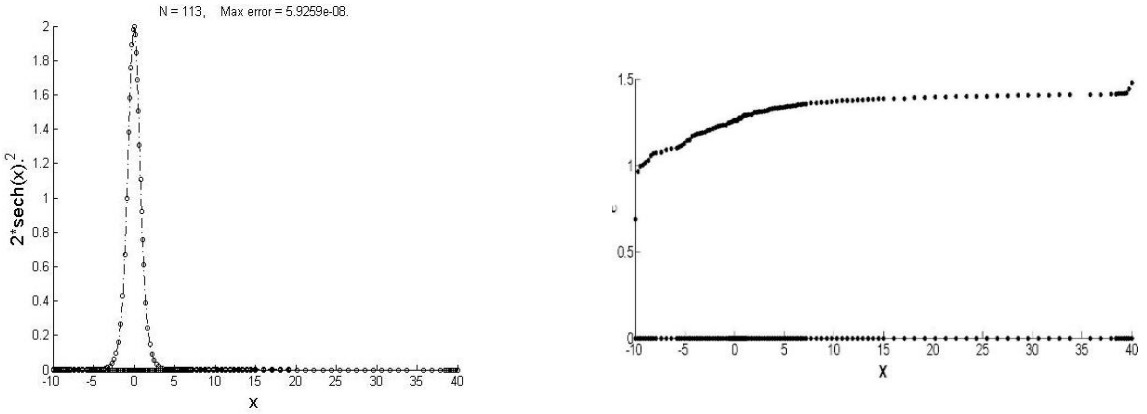
$$f_1(x) = 2sech^2(x); x \in [-10, 40], \tag{6.1}$$

$$f_2(x) = arc\tan(5x); x \in [-2,2], \tag{6.2}$$

$$f_3(x) = e^{x^3} + \cos(2x); x \in [0,1]. \tag{6.3}$$

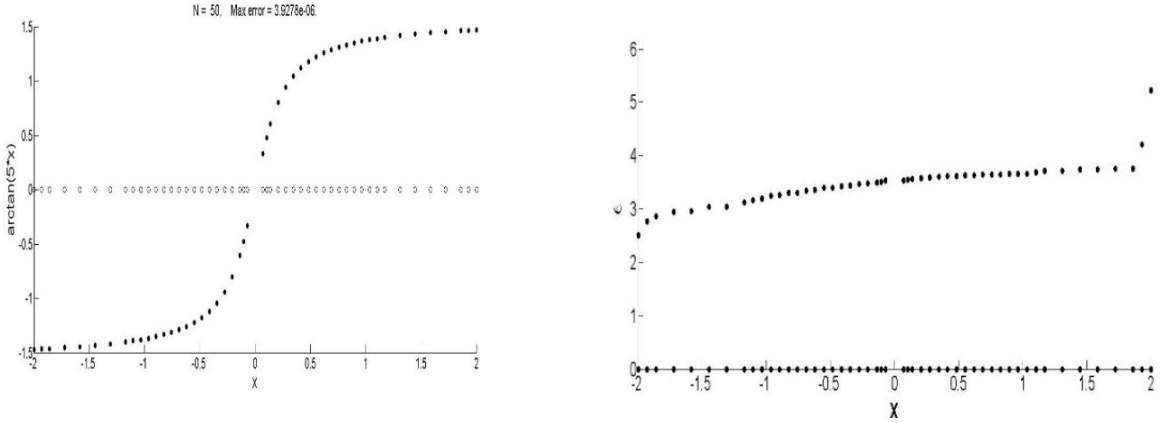
In the numerical problems (6.1), (6.2) and (6.3) multi quadric RBFs are used to interpolate the functions. For the interpolation, functions with steep gradient are chosen. The plots in Figs 1, 2 and 3 of the functions f_1, f_2 and f_3 show how the nodes distributed themselves near the steep gradient.

To estimate the optimal variable scaling parameter and the optimal number of centers for the function f_1, we choose the interval $R_\epsilon = [0.5, 2]$ which was proposed in [23]. The application of our proposed algorithm to the function f_1 uses only 113 nodes to produce the desired accuracy, whereas in [23] the same function f_1 uses *200* nodes to give the same accuracy. So, our proposed method uses nearly half of the centers to produce desired accuracy.



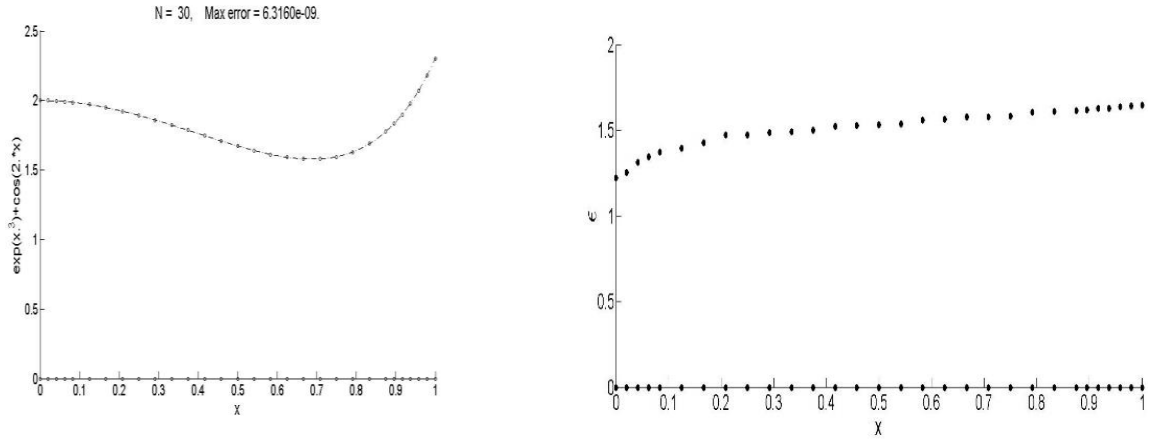
(a) (b)

Fig.1. Left: The function f_1 with the final RBF node distribution. Right: plot of X versus ϵ for f_1.



(a) (b)

Fig.2. Left: The function f_2 with the final RBF node distribution. Right: plot of X versus ϵ for f_2.



(a) (b)

Fig.3. Left: The function f_3 with the final RBF node distribution. Right: plot of X versus ϵ for f_3.

For the function f_2 the proposed interval was $R_\epsilon = [1.5, 5]$, the number of centers used by our proposed algorithm was *50* and to get the desired accuracy, the number of centers used in [23] was *60*. In this case also the proposed algorithm takes only half of the centers. Similarly, for the function f_3 the interval was $R_\epsilon = [4,7]$, the centers used in [23] were *60* and our algorithm uses only *30* centers to produce the same accuracy. Table 1 shows the numerical results for the functions f_1, f_2 and f_3.

Table 1. The results of the proposed method for the functions f_1, f_2 and f_3.

Function	R_ε	Number of centers (N)		Max. error	
		Non-Adaptive [23]	Adaptive (Present method)	Non-Adaptive [23]	Adaptive (Present method)
f_1	[*0.5, 2*]	*200*	*113*	3.85×10^{-8}	5.92×10^{-8}
f_2	[*1.5, 5*]	*60*	*50*	3.21×10^{-6}	3.92×10^{-6}
f_3	[*4, 7*]	*60*	*30*	1.08×10^{-8}	6.31×10^{-9}

7. Conclusions and future work

In this paper, we have presented the adaptive residual subsampling method (ARSM) [1] along with an optimization [24, 25] algorithm and then tested it for one dimensional RBF interpolation. The proposed method produces an optimal variable shape parameter and also an optimal number of centers according to the desired accuracy. From the numerical examples, we have observed that the number of centers used by the proposed method was almost *50* percent less when compared with the centers used in [23] to get the desired accuracy. In future, we would like to extend ARSM with an optimization algorithm for steady and unsteady differential equations.

Nomenclature

ϵ – shape parameter

X – discrete set of centres for the function f

Y – set of test points

$\mathbf{\Omega}$ – domain of the function f

$\mathcal{U}$ – approximate function of f

R_ϵ – interval of the shape parameter

θ_r – upper threshold

θ_c – lower threshold

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