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COMPUTATIONAL SCHEME FOR A DIFFERENTIAL – DIFFERENCE EQUATION WITH A LARGE DELAY IN CONVECTION TERM

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A computational scheme for the solution of layer behaviour differential equation involving a large delay in the derivative term is devised using numerical integration. If the delay is greater than the perturbation parameter, the layer structure of the solution is no longer preserved, and the solution oscillates. A numerical method is devised with the support of a specific kind of mesh in order to reduce the error and regulate the layered structure of the solution with a fitting parameter. The scheme is discussed for convergence. The maximum errors in the solution are tabulated and compared to other methods in the literature to verify the accuracy of the numerical method. Using this specific kind of mesh with and without the fitting parameter, we also studied the layer and oscillatory behavior of the solution with a large delay.

Key words: delay differential equation, layer behavior, large delay, fitting parameter.

1. Introduction

The delay argument differential equations of layer behaviour examined in this research are particularly relevant to the theoretical investigation of neural variability. Readers interested in learning more about the complex models of nerve membrane potential with stochastic synaptic input and a comprehensive discussion of these challenges might look into the works of Derstine *et al.* [1], Kuang [2], Mackey and Glass [3] and Stein [4].

A model for the stochastic effects resulting from neuronal variability was proposed by Stein [4] to used Monte Carlo techniques to approximate the solution to the differential – difference equation (DDE) model. Although the theory and numerical approximation to delay differential equations has been widely researched in the literature (see [5], [6], [7], [8] and the references therein), singularly perturbed delay differential equations (SPDDEs) which are also called layer behaviour differential equations with small shifts have received less attention. The authors in [9, 10, 11, 12] produced variety of numerical methods to solve a class of singular perturbation problems. However, there are some studies published [13-15] in which finite differences on piecewise-uniform meshes were employed to solve SPDDEs. The basic limitation of all of these papers is that the study is limited to small shift arguments, i.e., when the delay term is small and is of the order of the diffusion coefficient, and so all studies are based on the Taylor series expansion approach.

Lange and Miura [16, 17] address problems with solutions that possess layer behaviour at one or both boundaries in addition to oscillatory structure. The application of Laplace transforms to the layer equations obtains unexpected results. It is demonstrated that as the shifts increase, the layer behaviour can change and even be destroyed.

Bansal and Sharma [18] studied a novel fitted operator method to solve a time dependent convection diffusion reaction problem with mixed large shifts. Das [19] derived an evaluation of the parameter uniform a deductive error of a nonlinear system of SPDDE with both layers in every component of the solution.

Ravi Kanth and Murali [20] proposed a hybrid technique using a tension spline to solve a class of nonlinear convection delay problems. The same authors in [21] used a tension spline with fitting factor to solve

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a diffusion equation with convection delayed dominated. The same tension spline is used to solve a time dependent SPDE. Kumara Swamy et. al. [22] devised a difference scheme of fourth order with a fitting factor to solve a SPDDEs with mixed shifts. Lalu *et al.* [23] suggested a numerical approach using a non-polynomial spline to solve a layer behaviour DDEs with small and large delay in the differentiated term. In [24], the authors derived an accelerated fitted scheme for solving SPDE with non-local boundary condition.

Motivated by previous efforts, we propose in this article a numerical method for SPDE that works for both small and large shift argument without resorting to the Taylor series expansion.

2. Problem description

Consider a layer behavior delay differential equation:

$$\varepsilon\nu''(s)+g(s)\nu'(s-\delta)+p(s)\nu(s)=\mathcal{F}(s), \qquad 0<s<1, \tag{2.1}$$

with the conditions

$$\nu(0)=\alpha \quad \text{and} \quad \nu(1)=\beta \tag{2.2}$$

where $\varepsilon(0<\varepsilon<<1)$ is the perturbation parameter, the delay parameter is δ , the functions $g(s),p(s),\mathcal{F}(s)$ are bounded in $(0,1)$ and α,β are the finite constants. Equation (2.1) reveals layers and turning points when ε is small, depending on the term of the convection coefficient. For δ is of $o(\varepsilon)$ and $\delta\neq 0$, the layer performance of the solution of Eq. (2.1) is preserved. If the delay is larger than the perturbation, the solution oscillates due to the loss of layer behaviour.

2.1. Layer at left – end

If $g(s)\geq P>0$ on $s_i\leq s\leq s_{i+1}$, P is a positive constant, Eq.(2.1) has the solution with a layer at the left end of the domain $s_i\leq s\leq s_{i+1}$.
With the Taylor expansion, we have:

$$\nu'(s-\varepsilon)=\nu'(s)-\varepsilon\nu''(s),$$

which gives $\quad \varepsilon\nu''(s)=\nu'(s)-\nu'(s-\varepsilon) \quad$ (2.3)
Utilizing Eq. (2.3) in Eq. (2.1), we get the first order DDE

$$\nu'(s)=\nu'(s-\varepsilon)-g(s)\nu'(s-\delta)-p(s)\nu(s)+\mathcal{F}(s). \tag{2.4}$$

The interval $[0,1]$ is divided into N equal domain of step size $h=\frac{1}{N}$ so that $s_i=ih,\ i=0,1,\dots,N$ are the grid points. The delay term is handled by the step size as $h=\frac{\delta}{m}$, $m=rq$, and q is the mantissa of δ and r is a positive integer. So that the term $\nu'(s-\delta)$ becomes a mesh point.
Integration of Eq. (2.4) with respect to s between s_i and s_{i+1} gives:

$$\int_{s_i}^{s_{i+1}} \nu'(s)\,ds = \int_{s_i}^{s_{i+1}} \left[\nu'(s-\varepsilon) - g(s)\nu'(s-mh) - p(s)\nu(s) + \mathcal{F}(s)\right]ds,$$

$$\nu_{i+1} - \nu_i = \int_{s_i}^{s_{i+1}} \left[\nu'(s-\varepsilon) - g(s)\nu(s-mh) - p(s)\nu(s) + \mathcal{F}(s)\right]ds, \tag{2.5}$$

$$\nu_{i+1} - \nu_i = \nu(s_{i+1}-\varepsilon) - \nu(s_i-\varepsilon) - g(s_{i+1})\nu(s_{i+1}-mh) + p(s_i)\nu(s_i - mh) +$$
$$+ \int_{s_i}^{s_{i+1}} \left(g'(s)\nu(s-mh) - p(s)\nu(s) + \mathcal{F}(s)\right)ds.$$

Applying the trapezoidal rule for the integral term, we get:

$$\nu_{i+1} - \nu_i = \nu(s_{i+1}-\varepsilon) - \nu(s_i-\varepsilon) - g(s_{i+1})\nu(s_{i+1}-mh) + g(s_i)\nu(s_i - mh) +$$
$$+\frac{h}{2}\left(g'(s_i)\nu_{i-m} + g'(s_{i+1})\nu_{i-m+1}\right) - \frac{h}{2}\left(p_{i+1}\nu_{i+1} + p_i\nu_i\right) + \frac{h}{2}\left(\mathcal{F}_{i+1} + \mathcal{F}_i\right) = 0.$$

Using linear interpolation, the above equation is simplified to;

$$\left(\nu_{i+1}\left(1-\frac{\varepsilon}{h}\right) + \varepsilon\left(\frac{\nu_i}{h}\right)\right) - \left(\nu_i\left(1-\frac{\varepsilon}{h}\right) + \varepsilon\left(\frac{\nu_{i-1}}{h}\right)\right) - g(s_{i+1})\nu_{i-m+1} + g(s_i)\nu_{i-m} +$$
$$+\frac{h}{2}\left(g'(s_i)\nu_{i-m} + g'(s_{i+1})\nu_{i-m+1}\right) - \frac{h}{2}\left(p_{i+1}\nu_{i+1} + p_i\nu_i\right) + \frac{h}{2}\left(\mathcal{F}_{i+1} + \mathcal{F}_i\right) = \nu_{i+1} - \nu_i,$$

$$\left(\nu_{i+1}\left(1-\frac{\varepsilon}{h}\right) + \varepsilon\left(\frac{\nu_i}{h}\right)\right) - \left(\nu_i\left(1-\frac{\varepsilon}{h}\right) + \varepsilon\left(\frac{\nu_{i-1}}{h}\right)\right) - g(s_{i+1})\nu_{i-m+1} + g(s_i)\nu_{i-m} +$$
$$+\frac{h}{2}\left(g'(s_i)\nu_{i-m}\right) + \frac{h}{2}\left(g'(s_{i+1})\nu_{i-m+1}\right) - \frac{h}{2}p_{i+1}\nu_{i+1} - \frac{h}{2}p_i\nu_i + \frac{h}{2}\left(\mathcal{F}_{i+1} + \mathcal{F}_i\right) = \nu_{i+1} - \nu_i, \tag{2.6}$$

$$\left(\frac{\varepsilon}{h} + \frac{h\,p_{i+1}}{2}\right)\nu_{i+1} + \left(-\frac{2\varepsilon}{h} + \frac{h\,p_i}{2}\right)\nu_i + \left(\frac{\varepsilon}{h}\right)\nu_{i-1} +$$
$$+\left(g_{i+1} - \frac{hg'_{i+1}}{2}\right)\nu_{i-m+1} - \left(g_i + \frac{hg'_i}{2}\right)\nu_{i-m} + \left(G_i\right)\nu_{i-m-1} = \frac{h}{2}\left(\mathcal{F}_i + \mathcal{F}_{i+1}\right).$$

To handle the layer character, a fitting parameter $\sigma(\rho)$ is included in Eq. (2.6). The value of $\sigma(\rho)$ is calculated using the singular perturbations theory and is given by

$$\sigma(\rho) = \rho\frac{e^{g(0)\mathrm{m}\rho}\, e^{-\frac{g\rho}{2}}}{\left(e^{\frac{g\rho}{2}} - e^{-\frac{g\rho}{2}}\right)} \qquad \text{where} \qquad \rho = \frac{\varepsilon}{h},$$

$$\left(\frac{\sigma\varepsilon}{h}+\frac{h\,p_{i+1}}{2}\right)\nu_{i+1}+\left(-\frac{2\varepsilon}{h}+\frac{h\,p_i}{2}\right)\nu_i+\left(\frac{\sigma\varepsilon}{h}\right)\nu_{i-1}+$$
$$\left(g_{i+1}-\frac{hg'_{i+1}}{2}\right)\nu_{i-m+1}+\nu_{i-m}+\left(G_i\right)\nu_{i-m-1}=\frac{h}{2}\left(\mathcal{F}_i+\mathcal{F}_{i+1}\right),$$

$$L_i\nu_{i+1}+C_i\nu_i+R_i\nu_{i-1}+D_i\nu_{i-m+1}+E_i\nu_{i-m}+G_i\nu_{i-m-1}=H_i \quad \text{for} \quad i = 1(1)\, N\text{-}1. \tag{2.7}$$

$$L_i=\left(\frac{\sigma\varepsilon}{h}+\frac{h\,p_{i+1}}{2}\right),\quad C_i=\left(-\frac{2\varepsilon}{h}+\frac{h\,p_i}{2}\right),\quad R_i=\left(\frac{\sigma\varepsilon}{h}\right),$$

$$D_i=\left(g_{i+1}-\frac{hg'_{i+1}}{2}\right),\quad E_i=-\left(g_i+\frac{hg'_i}{2}\right),\quad G_i=0,\quad H_i=\frac{h}{2}\left(\mathcal{F}_i+\mathcal{F}_{i+1}\right).$$

The system Eq. (2.7) can be written as:

$$\begin{aligned}
&L_i\nu_{i+1}+M_i\nu_i+R_i\nu_{i-1}=H_i-D_i\nu_{i-m+1}-E_i\nu_{i-m}-G_i\nu_{i-m-1} \quad \forall\ i=1 \text{ to } m-1,\\
&L_i\nu_{i+1}+M_i\nu_i+R_i\nu_{i-1}=H_i-D_i\nu_{i-m+1}-E_i\nu_{i-m}-G_i\nu_{i-m-1} \quad \text{for } i=m,\\
&L_i\nu_{i+1}+M_i\nu_i+R_i\nu_{i-1}+D_i\nu_{i-m+1}+E_i\nu_{i-m}=H_i-G_i\nu_{i-m-1} \quad \text{for } i=m+1,\\
&L_i\nu_{i+1}+M_i\nu_i+R_i\nu_{i-1}+D_i\nu_{i-m+1}+E_i\nu_{i-m}+G_i\nu_{i-m-1}=H_i \quad \forall\ i=m+2 \text{ to } N-1.
\end{aligned} \tag{2.8}$$

Partial pivoting in Gauss elimination is used to solve these equations.

2.2. Layer at right end

When $g(s)\le Q<0$ on $s_{i-1}\le s\le s_i$, Q is a negative constant, then Eq.(2.1) solution depicts the layer at the right end of the domain $s_{i-1}\le s\le s_i$.
Utilizing the Taylor series, we have:

$$\nu'(s+\varepsilon)=\nu'(s)+\varepsilon\nu''(s),$$

which gives (2.9)

$$\varepsilon\nu''(s)=\nu'(s+\varepsilon)-\nu'(s).$$

With Eq. (2.9), Eq. (2.1) can be written as DDE of first order

$$\nu'(s)=\nu'(s+\varepsilon)+g(s)\nu'(s-\delta)+p(s)\nu(s)-\mathcal{F}(s). \tag{2.10}$$

Equation (2.4) integrated with respect to s from s_{i-1} to s_i, then we get:

$$\int_{s_{i-1}}^{s_i} \nu'(s)\,ds = \int_{s_{i-1}}^{s_i} (\nu'(s+\varepsilon)+g(s)\nu'(s-mh)+p(s)\nu(s)-\mathcal{F}(s)\,ds\,,$$

$$\nu_i-\nu_{i-1}=\int_{s_i-1}^{s_i}\left(v'(s+\varepsilon)+g(s)v^{'}(s-mh)+p(s)v(s)-\mathcal{F}(s)\right)ds\,, \tag{2.11}$$

$$\nu_i-\nu_{i-1}=\nu(s_i+\varepsilon)-\nu(s_{i-1}+\varepsilon)+g(s_i)\nu(s_i-mh)-g(s_{i-1})\nu(s_{i-1}-mh)+$$
$$+\int_{s_{i-1}}^{s_i}\left(g'(s)\nu(s-mh)+p(s)\nu(s)-\mathcal{F}(s)\right)ds.$$

Applying the trapezoidal rule for the integral term, we get:

$$\nu_i-\nu_{i-1}=\nu(s_i+\varepsilon)-\nu(s_{i-1}+\varepsilon)+g(s_i)\nu(s_i-mh)-g(s_{i-1})\nu(s_{i-1}-mh)+$$
$$+\frac{h}{2}\left(g'(s_i)\nu_{i-m}+g'(s_{i-1})\nu_{i-m-1}\right)+\frac{h}{2}\left(p_i\nu_i+p_{i-1}\nu_{i-1}\right)-\frac{h}{2}\left(\mathcal{F}_i+\mathcal{F}_{i-1}\right).$$

Using linear interpolation, above equation is simplified to

$$v_i-v_{i-1}=\left(v_i\left(1-\frac{\varepsilon}{h}\right)+\varepsilon\left(\frac{v_{i+1}}{h}\right)\right)-\left(v_{i-1}\left(1-\frac{\varepsilon}{h}\right)+\varepsilon\left(\frac{v_i}{h}\right)\right)-c(s_{i-1})v_{i-m-1}+$$
$$+c(s_i)v_{i-m}+\frac{h}{2}\left(c'(s_i)v_{i-m}+c'(s_{i+1})v_{i-m-1}\right)+\frac{h}{2}\left(d_iv_i+d_{i-1}v_{i-1}\right)-\frac{h}{2}\left(\mathcal{F}_i+\mathcal{F}_{i-1}\right),$$

$$\begin{aligned}&\left(\frac{\varepsilon}{h}\right)v_{i+1}+\left(-\frac{2\varepsilon}{h}-\frac{h}{2}d_i\right)v_i+\left(\frac{\varepsilon}{h}-\frac{h}{2}d_{i-1}\right)v_{i-1}+\left(-c_{i-1}+\frac{h}{2}c'_{i-1}\right)v_{i-m-1}+\\&+\left(c_i+\frac{h}{2}c'_i\right)v_{i-m}+\left(G_i\right)v_{i-m+1}-\frac{h}{2}\left(\mathcal{F}_i+\mathcal{F}_{i-1}\right)=0,\end{aligned} \tag{2.12}$$

$$\left(\frac{\varepsilon}{h}\right)v_{i+1}+\left(-\frac{2\varepsilon}{h}-\frac{h}{2}d_i\right)v_i+\left(\frac{\varepsilon}{h}-\frac{h}{2}d_{i-1}\right)v_{i-1}+\left(-c_{i-1}+\frac{h}{2}c'_{i-1}\right)v_{i-m-1}+$$
$$+\left(c_i+\frac{h}{2}c'_i\right)v_{i-m}+\left(G_i\right)v_{i-m+1}=\frac{h}{2}\left(\mathcal{F}_i+\mathcal{F}_{i-1}\right).$$

To handle the layer character, a fitting parameter $\sigma(\rho)$ is included in Eq. (2.12). The value of the fitting factor is obtained using the singular perturbation theory and is given by

$$\sigma(\rho)=\rho\frac{e^{g(0)m\rho}\ e^{\frac{g\rho}{2}}}{\left(e^{\frac{g\rho}{2}}-e^{-\frac{g\rho}{2}}\right)} \qquad \text{where} \qquad \rho=\frac{\varepsilon}{h}.$$

Then, Eq. (2.12) becomes:

$$\left(\frac{\sigma\varepsilon}{h}\right)v_{i+1}+\left(-\frac{2\varepsilon\sigma}{h}-\frac{h}{2}d_i\right)v_i+\left(\frac{\varepsilon\sigma}{h}-\frac{h}{2}d_{i-1}\right)v_{i-1}+$$
$$+\left(-c_{i-1}+\frac{h}{2}c'_{i-1}\right)v_{i-m-1}+\left(c_i+\frac{h}{2}c'_i\right)v_{i-m}+\left(G_i\right)v_{i-m+1}=\frac{h}{2}\left(\mathcal{F}_i+\mathcal{F}_{i-1}\right),$$

$$L_i v_{i+1}+C_i v_i+R_i v_{i-1}+D_i v_{i-m-1}+E_i v_{i-m}+G_i v_{i-m+1}=H_i,\quad i=1 \text{ to } N-1. \tag{2.13}$$

$$L_i=\left(\frac{\sigma\varepsilon}{h}\right),\quad C_i=\left(-\frac{2\varepsilon\sigma}{h}-\frac{h\,p_i}{2}\right),\quad R_i=\left(\frac{\varepsilon\sigma}{h}-\frac{h\,p_{i-1}}{2}\right),$$

$$D_i=\left(-g_{i-1}+\frac{hg'_{i-1}}{2}\right),\quad E_i=\left(g_i+\frac{hg'_i}{2}\right),\quad G_i=0,\ H_i=\frac{h}{2}\left(\mathcal{F}_i+\mathcal{F}_{i-1}\right).$$

The system of Eq.(2.13) can be written as:

$$\begin{aligned}
&L_i v_{i+1}+C_i v_i+R_i v_{i-1}=H_i-G_i v_{i-m+1}-E_i v_{i-m}-D_i v_{i-m-1} \quad \forall\, 1\le i\le m-1,\\
&L_i v_{i+1}+C_i v_i+R_i v_{i-1}+G_i v_{i-m+1}=H_i-E_i v_{i-m}-D_i v_{i-m-1} \quad \text{for } i=m,\\
&L_i v_{i+1}+C_i v_i+R_i v_{i-1}+G_i v_{i-m+1}+E_i v_{i-m}=H_i-D_i v_{i-m-1} \quad \text{for } i=m+1,\\
&L_i v_{i+1}+C_i v_i+R_i v_{i-1}+G_i v_{i-m+1}+E_i v_{i-m}+D_i v_{i-m-1}=H_i \quad \forall\ m+2\le i\le N-1.
\end{aligned} \tag{2.14}$$

Partial pivoting in Gauss elimination is used to solve the above system.

3. Analysis of convergence

The matrix form of the system Eq. (2.7) together with the boundary constraints is

$$\mathcal{A}n+\mathcal{B}+T_i\left(h\right)=0, \tag{3.1}$$

$$\mathcal{A}=\begin{bmatrix} C_1 & R_1 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\ L_2 & C_2 & R_2 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\ 0 & L_3 & C_3 & R_3 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ D_m & 0 & \cdots & L_m & C_m & R_m & 0 & \cdots & \cdots & \cdots & 0 \\ E_{m+1} & D_{m+1} & 0 & \cdots & L_{m+1} & C_{m+1} & R_{m+1} & 0 & \cdots & \cdots & 0 \\ G_{m+2} & E_{m+2} & D_{m+2} & 0 & \cdots & L_{m+2} & C_{m+2} & R_{m+2} & 0 & \cdots & 0 \\ 0 & G_{m+3} & E_{m+3} & D_{m+3} & 0 & \cdots & L_{m+3} & C_{m+3} & R_{m+3} & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & 0 \\ \cdots & \cdots & 0 & G_{N-2} & E_{N-2} & D_{N-2} & 0 & \cdots & L_{N-2} & C_{N-2} & R_{N-2} \\ \cdots & \cdots & \cdots & 0 & G_{N-1} & E_{N-1} & D_{N-1} & 0 & \cdots & L_{N-1} & C_{N-1} \end{bmatrix}$$

and $\mathcal{B}=\left[r_1, r_2, r_3,\ldots, r_m, r_{m+1}, r_{m+2},\ldots, r_{N-2}, r_{N-1}\right]$

here
$$r_i=\begin{cases} H_i - D_i v_{i-m+1} - E_i v_{i-m} - G_i v_{i-m-1} - L_1 v_0 & \forall \quad i=1 \text{ to } m-1, \\ H_i - E_i v_{i-m} - G_i v_{i-m-1} & \text{for } \; i=m, \\ H_i - G_i v_{i-m-1} & \text{for } \; i=m+1, \\ H_i - C_{N-1} v_N & \forall \quad i=m+2 \text{ to } N-1 \end{cases}$$

where
$$L_i=\left(\frac{\sigma\varepsilon}{h}+\frac{h\,p_{i+1}}{2}\right),\quad C_i=\left(-\frac{2\varepsilon}{h}+\frac{h\,p_i}{2}\right),\quad R_i=\left(\frac{\sigma\varepsilon}{h}\right),$$

$$D_i=\left(g_{i+1}-\frac{hg'_{i+1}}{2}\right),\quad E_i=-\left(g_i+\frac{hg'_i}{2}\right),\quad G_i=0,\ H_i=\frac{h}{2}\left(\mathcal{F}_i+\mathcal{F}_{i+1}\right).$$

The local truncation in the scheme is

$$T_i(h)\le h^3 g_i m v_i'' \text{ i.e., } T_i(h)\le o\left(h^3\right)$$

$n=\left[n_1,n_2,\ldots,n_{N-1}\right]^T, T_i(h)=\left[T_1,T_2,\ldots,T_{N-1}\right]^T,\ O=\left[0,0,\ldots,0\right]^T$ which are related vectors of Eq. (2.3).
Let $s=\left[s_1,s_2,\ldots,s_{N-1}\right]^T \cong s$ satisfy the equation.

$$\mathcal{A}s+\mathcal{B}=0. \tag{3.2}$$

Let $e_i=n_i-s_i, i=1(1)N-1$ be the error so that $E=\left[e_1,e_2,\ldots,e_{N-1}\right]^T=n-s$.
Using Eq. (3.1) and Eq. (3.2), the error equation is:

$$\mathcal{A}E=T_i(h) \tag{3.3}$$

Let $\tilde{S}_i$ be the sum of the i^{th} row elements of the matrix $\mathcal{A}$. Then we have:

$$\tilde{S}_i = L_i + C_i + R_i = \varepsilon + \frac{h^2 p_{i+1}}{2} - 2\varepsilon + \frac{h^2 p_i}{2} + \varepsilon = \frac{h^2}{2}\left(p_{i+1} + p_i\right) \qquad \forall\ i = 1 \text{ to } m-1,$$

$$\begin{aligned}\tilde{S}_i &= L_i + C_i + R_i + D_i = \frac{h^2}{2}\left(p_{i+1} - p_i\right) + hp_{i+1} + \frac{h^2}{2}\left(g'_{i+1}\right) = \\ &= hg_{i+1} + \frac{h^2}{2}\left(p_{i+1} + p_i - g'_{i+1}\right)\end{aligned} \qquad \text{for } i = m,$$

$$\begin{aligned}\tilde{S}_i &= L_i + C_i + R_i + D_i + E_i = hg_{i+1} + \frac{h^2}{2}\left(p_{i+1} + p_i - g'_{i+1}\right) - \frac{h^2}{2} g'_i - hg_i = \\ &= h\left(g_{i+1} - g_i\right) + \frac{h^2}{2}\left(p_{i+1} + p_i - g'_{i+1} - g'_i\right),\end{aligned} \qquad \text{for } i = m+1.$$

$$\begin{aligned}\tilde{S}_i &= L_i + C_i + R_i + D_i + E_i + G_i = \\ &= h\left(g_{i+1} - g_i\right) + \frac{h^2}{2}\left(p_{i+1} + p_i - g'_{i+1} - g'_i\right)\end{aligned} \qquad \forall\ m+2 \le i \le N-1.$$

Let $\zeta_{1^*} = \min_{1\le i\le N}\left|g\left(x_i\right)\right|$ and $\zeta_1^* = \max_{1\le i\le N}\left|g\left(x_i\right)\right|$, $\zeta_{2^*} = \min_{1\le i\le N}\left|p\left(x_i\right)\right|$ and $\zeta_2^* = \max_{1\le i\le N}\left|p\left(x_i\right)\right|$.
Since $0 < \varepsilon \ll 1$ and $\varepsilon \propto o\left(h\right)$, $\mathcal{A}$ is irreducible and monotone with a small h [25, 26]. Therefore $\mathcal{A}^{-1}$ exists and $\mathcal{A}^{-1} \ge 0$. So, we can deduce from Eq.(3.2) that:

$$\left\|E\right\| \le \left\|\mathcal{A}^{-1}\right\|\left\|T\right\|. \tag{3.4}$$

For sufficiently small h, we have;

$$\tilde{S}_i \ge \frac{h^2}{2}\left(p_{i+1} + p_i\right) \ge h^2, \quad \forall\ \ i = 1 \text{ to } m-1,$$

$$\tilde{S}_i \ge hc_{i+1} + \frac{h^2}{2}\left(d_{i+1} + d_i - c'_{i+1}\right) \ge h^2, \quad \text{for } i = m,$$

$$\tilde{S}_i \ge h\left(c_{i+1} - c_i\right) + \frac{h^2}{2}\left(d_{i+1} + d_i - c'_{i+1} - c'_i\right) \ge h^2, \quad \text{for } i = m+1,$$

$$\tilde{S}_i \ge h\left(c_{i+1} - c_i\right) + \frac{h^2}{2}\left(d_{i+1} + d_i - c'_{i+1} - c'_i\right) \ge h^2, \quad \forall\ m+2\ (1) i \le N-1.$$

Let $\mathcal{A}_{i,k}^{-1}$ be the$(i,k)^{th}$ element of $\mathcal{A}^{-1}$ and define $\left\|\mathcal{A}^{-1}\right\| = \max_{1\le i\le N-1} \sum_{k=1}^{N-1} \mathcal{A}_{i,k}^{-1}$ and $\left\|T\left(h\right)\right\| = \max_{1\le i\le N-1}\left|T_i\left(h\right)\right|$.

Since $\mathcal{A}_{i,k}^{-1} \geq 0$ and $\sum_{k=1}^{N-1} \mathcal{A}_{i,k}^{-1} \cdot \tilde{\mathcal{S}}_k = 1$ for $\forall\ i = 1$ to $N-1$.

Hence
$$\sum_{k=1}^{m-1} \mathcal{A}_{i,k}^{-1} \leq \frac{1}{\min\limits_{1 \leq k \leq m-1} \tilde{\mathcal{S}}_k} < \frac{1}{h^2 \zeta_{2^*}}, \quad \forall\ i = 1 \text{ to } m-1. \tag{3.5}$$

$$\mathcal{A}_{i,k}^{-1} \leq \frac{1}{\tilde{\mathcal{S}}_m} < \frac{1}{h^2}, \quad i = m, m+1. \tag{3.6}$$

Furthermore

$$\sum_{k=m+2}^{N-1} \mathcal{A}_{i,k}^{-1} \leq \frac{1}{\min\limits_{1 \leq k \leq m-1} \tilde{\mathcal{S}}_k} < \frac{1}{h^2}, \quad i = m+2 \text{ to } N-1. \tag{3.7}$$

Using Eqs. (4.5)-(4.7) and (4.4), we get:

$$\|E\| \leq O(h). \tag{3.8}$$

Hence, the proposed scheme is first-order convergent.

4. Numerical illustrations

To show the relative efficiency of the proposed approach, it is tested numerically on the following examples for various values of delay. The maximum absolute errors (MAEs) in the solution are calculated using the double mesh principle $E^N = \max\limits_{0 \leq i \leq N} \left| v_i^N - W_{2i}^{2N} \right|$. The MAEs are organized in the form of Tabs (1-6).

Example 1. $\varepsilon v''(s) + v'(s-\delta) - v(s) = 0$ with $v(0) = 1$ and $v(1) = 1$.

Example 2. $\varepsilon v''(s) - v'(s-\delta) - v(s) = 0$ with $v(0) = 1$ and $v(1) = -1$.

Example 3. $\varepsilon v''(s) + 0.25 v'(s-\delta) - v(s) = 0$ with $v(0) = 1$ for $-\delta \leq s \leq 0$, $v(1) = 1$.

5. Conclusion

In this article, a computational technique for the solution of SPDE with a large delay is derived. Designing the mesh size to mange the delay argument, a numerical strategy with a fitting parameter is devised to reduce the error and manage the oscillations in the solution. Three examples are solved and computational results with large delay are tabulated in Tabs (1-4). It is revealed that as the size of the mesh decreses, the maximum error reduces. The layer description in the solutions with and without the fitting parameter is shown in the Figs (4-6). We identified that at large delay values, the fitting parameter regulates the oscillations in the layer.

Table 1. The MAE in Example 1 for $\varepsilon = 10^{-2}$.

N	$\delta = 1\times\varepsilon$	$\delta = 2\times\varepsilon$	$\delta = 3\times\varepsilon$	$\delta = 4\times\varepsilon$	$\delta = 5\times\varepsilon$
500	1.4870e-02	1.4835e-02	1.2596e-02	8.8433e-03	4.1954e-03
1000	7.3741e-03	7.2802e-03	6.1747e-03	4.2722e-03	2.0044e-03
1500	4.8997e-03	4.8234e-03	4.0892e-03	2.8153e-03	1.3162e-03
2000	3.6682e-03	3.6063e-03	3.0566e-03	2.0993e-03	9.7967e-04
2500	2.9317e-03	2.8797e-03	2.4404e-03	1.6736e-03	7.8019e-04
3000	2.4416e-03	2.3967e-03	2.0309e-03	1.3914e-03	6.4819e-04

Table 2. The MAE in Example 2 for $\varepsilon = 10^{-2}$.

N	$\delta = 1\times\varepsilon$	$\delta = 2\times\varepsilon$	$\delta = 3\times\varepsilon$	$\delta = 4\times\varepsilon$	$\delta = 5\times\varepsilon$
500	1.2630e-02	7.9524e-03	5.7670e-03	4.4710e-03	4.5794e-03
1000	6.3804e-03	3.9864e-03	2.8827e-03	2.2332e-03	1.9333e-03
1500	4.2685e-03	2.6602e-03	1.9218e-03	1.4887e-03	1.2345e-03
2000	3.2070e-03	1.9963e-03	1.4414e-03	1.1165e-03	9.1091e-04
2500	2.5685e-03	1.5976e-03	1.1532e-03	8.9316e-04	7.2338e-04
3000	2.1420e-03	1.3316e-03	9.6106e-04	7.4428e-04	6.0060e-04

Table 3. The MAE in Example 3 for $\varepsilon = 10^{-2}$.

N	$\delta = 1\times\varepsilon$	$\delta = 2\times\varepsilon$	$\delta = 3\times\varepsilon$	$\delta = 4\times\varepsilon$	$\delta = 5\times\varepsilon$
500	1.7015e-03	3.3024e-03	4.1758e-03	4.5771e-03	4.6996e-03
1000	1.0093e-03	1.7533e-03	2.1528e-03	2.3291e-03	2.3744e-03
1500	7.0879e-04	1.1922e-03	1.4500e-03	1.5618e-03	1.5884e-03
2000	5.4523e-04	9.0285e-04	1.0930e-03	1.1748e-03	1.1934e-03
2500	4.4278e-04	7.2651e-04	8.7704e-04	9.4147e-04	9.5570e-04
3000	3.7264e-04	6.0776e-04	7.3234e-04	7.8547e-04	7.9696e-04

Table 4. The MAE in Example 1 for $\varepsilon = 10^{-1}$.

$N\downarrow$	$\delta = 0.03$	$\delta = 0.05$	$\delta = 0.08$
		without fitting parameter	
100	8.2833e-03	1.0435e-02	1.3445e-02
200	4.1220e-03	5.3042e-03	6.9441e-03
300	2.7475e-03	3.5604e-03	4.6869e-03
400	2.0611e-03	2.6795e-03	3.5370e-03
500	1.6492e-03	2.1488e-03	2.8406e-03
		with fitting parameter	
100	2.0553e-03	3.9714e-03	5.7023e-03
200	1.3835e-03	2.2396e-03	2.9922e-03
300	1.0045e-03	1.5497e-03	2.0260e-03
400	7.8370e-04	1.1835e-03	1.5313e-03
500	4.1169e-04	9.5705e-04	1.2308e-03

Cont.Table 4. The MAE in Example 1 for $\varepsilon = 10^{-1}$.

$N\downarrow$	$\delta = 0.03$	$\delta = 0.05$	$\delta = 0.08$
		Results in [24]	
100	*2.0790(-03)*	*5.3420(-03)*	*3.8340(-02)*
200	*1.1020(-03)*	*3.2160(-03)*	*2.1830(-02)*
300	*7.4900(-04)*	*2.2860(-03)*	*1.5200(-02)*
400	*5.6800(-04)*	*1.7700(-03)*	*1.1650(-02)*
500	*4.5700(-04)*	*1.4440(-03)*	*9.4430(-03)*

Table 5.The MAE in Example 2 for $\varepsilon = 10^{-1}$.

$N\downarrow$	$\delta = 0.03$	$\delta = 0.05$	$\delta = 0.08$
		without fitting parameter	
100	*1.0402e-03*	*2.1902e-03*	*3.0325e-03*
200	*5.1090e-04*	*1.2205e-03*	*1.8141e-03*
300	*3.3906e-04*	*8.4234e-04*	*1.2782e-03*
400	*2.5374e-04*	*6.4263e-04*	*9.8485e-04*
500	*2.0272e-04*	*5.1935e-04*	*8.0054e-04*
		with fitting parameter	
100	*2.4117e-04*	*2.4335e-04*	*2.4398e-04*
200	*6.1389e-05*	*6.1668e-05*	*6.1755e-05*
300	*2.7448e-05*	*2.7531e-05*	*2.7558e-05*
400	*1.5486e-05*	*1.5521e-05*	*1.5532e-05*
500	*9.9287e-06*	*9.9467e-06*	*9.9526e-06*
		Results in [24]	
100	*7.6000(-04)*	*1.1050(-03)*	*1.4140(-03)*
200	*3.8000(-04)*	*5.5200(-04)*	*7.0700(-04)*
300	*2.5300(-04)*	*3.6800(-04)*	*4.7200(-04)*
400	*1.9000(-04)*	*2.7600(-04)*	*3.5400(-04)*
500	*1.5200(-04)*	*2.2100(-04)*	*2.8300(-04)*

Table 6. The MAE in Example 3 for $\varepsilon = 10^{-1}$.

$N\downarrow$	$\delta = 0.03$	$\delta = 0.05$	$\delta = 0.08$
		without fitting parameter	
100	*2.4239e-03*	*2.4351e-03*	*2.4311e-03*
200	*6.9989e-04*	*7.2482e-04*	*9.2091e-04*
300	*3.5913e-04*	*3.8354e-04*	*6.3925e-04*
400	*2.2972e-04*	*2.6405e-04*	*4.8897e-04*
500	*1.6492e-04*	*2.2073e-04*	*3.9577e-04*

Cont.Table 6. The MAE in Example 3 for $\varepsilon = 10^{-1}$.

$N\downarrow$	$\delta = 0.03$	$\delta = 0.05$	$\delta = 0.08$
		with fitting parameter	
100	5.0655e-04	1.9715e-03	1.8372e-03
200	4.3136e-04	5.2986e-04	5.5667e-04
300	2.6804e-04	3.4666e-04	3.6534e-04
400	2.1012e-04	2.5759e-04	2.7189e-04
500	1.4065e-04	2.0493e-04	2.1651e-04
		Results in [24]	
100	7.6000(-04)	1.1050(-03)	1.4140(-03)
200	3.8000(-04)	5.5200(-04)	7.0700(-04)
300	2.5300(-04)	3.6800(-04)	4.7200(-04)
400	1.9000(-04)	2.7600(-04)	3.5400(-04)
500	1.5200(-04)	2.2100(-04)	2.8300(-04)

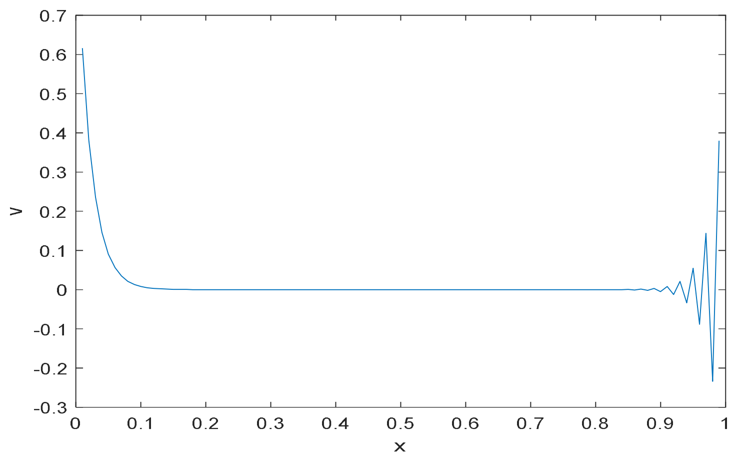

Fig.1. Layer profile in Example 1 for $\varepsilon = 0.001$ and $\delta = 5\varepsilon$ without the fitting parameter.

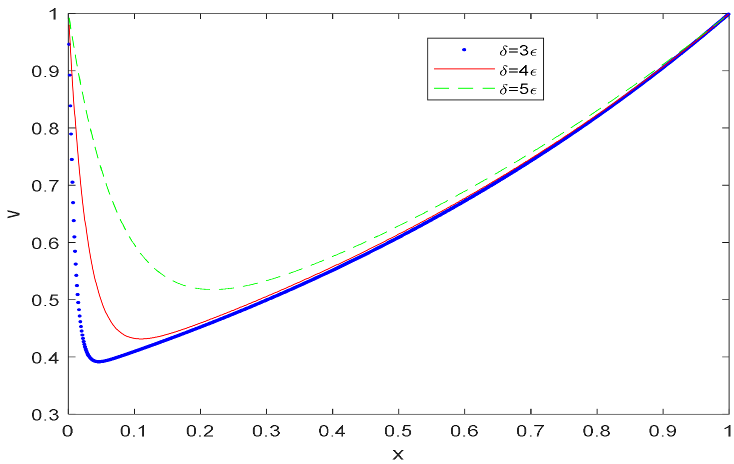

Fig.2. Layer profile in Example 1 for $\varepsilon = 0.001$ and $\delta = 5\varepsilon$ with the fitting parameter.

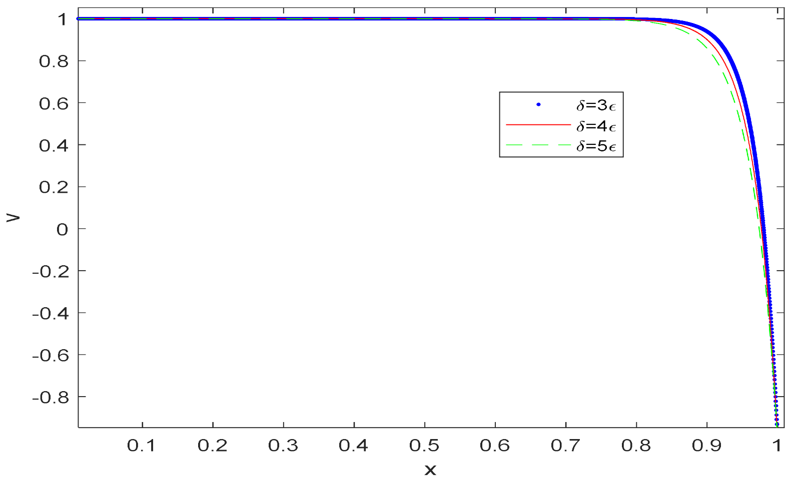

Fig.3. Layer profile in Example 2 for $\varepsilon = 0.001$ and $\delta = 5\varepsilon$ with the fitting parameter.

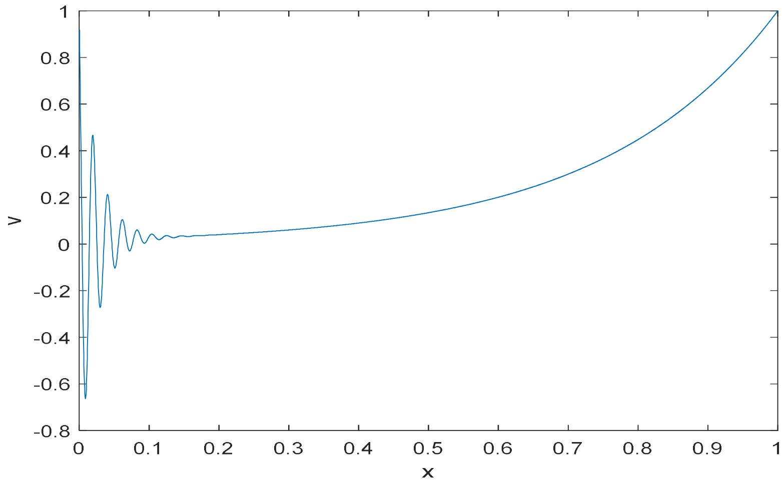

Fig.4. Layer profile in Example 3 for $\varepsilon = 0.001$ and $\delta = 5\varepsilon$ without the fitting parameter.

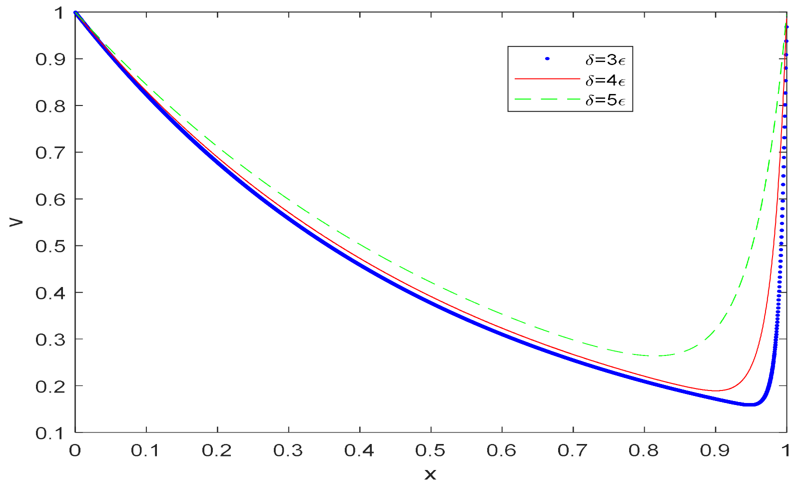

Fig.5. Layer profile in Example 3 for $\varepsilon = 0.001$ and $\delta = 5\varepsilon$ with the fitting parameter.

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Nomenclature

Nomenclature

C – positive constant
d – mantissa of δ
E – error matrix
E^N – absolute error
$g(x), p(x), \mathcal{F}(x)$ – smooth functions
h – mesh size
N – number of sub-intervals
P – positive constant
r – positive integer
$T_i(h)$ – truncation error
x – independent variable
x_i – mesh points
α, β – constants
δ – delay parameter
ε – perturbation parameter
v – solution
ρ – mesh ration
σ – fitting parameter

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