An Optimal Iterative Learning Control Approach for Linear Systems with Nonuniform Trial Lengths under Input Constraints

Zhihe Zhuang, Hongfeng Tao, Yiyang Chen, Vladimir Stojanovic, and Wojciech Paszke

***Abstract*—In the practical application of iterative learning control (ILC), the ILC process may end early by accident during the performance improvement, which yields the problems with nonuniform trial lengths. For such practical systems, input signals are usually constrained because of some physical limitation. Therefore, this paper proposes an optimal ILC algorithm for linear time-invariant multiple-input multiple-output (MIMO) systems with nonuniform trial lengths under input constraints. By introducing the primal-dual interior point method during the ILC design, the proposed algorithm actively embeds the input constraints into the ILC process. Moreover, the monotonic convergence of the proposed algorithm is derived theoretically in the sense of mathematical expectation, which further deduces a corollary under the circumstance that the desired input can not be obtained. The effectiveness of the proposed algorithm is verified on a numerical mobile robot simulation model.**

***Index Terms*—Iterative learning control (ILC), nonuniform trial lengths, input constraints, primal-dual interior point method.**

I. Introduction

ITERATIVE learning control (ILC) is an effective approach that uses previous experiment data to handle the repetitive control processes, including chemical batch processing [1], [2], industrial robotic systems [3], robotic-assisted biomedical systems [4], [5] and etc. Different from the traditional control technologies, ILC improves its control performance for a specific tracking task by learning from what has been done before, and thus the task should be repeatable over a fixed time interval [6]. More information can be referred through reviews on ILC like [7]–[9].

This work was supported in part by the National Natural Science Foundation of China under Grant 61773181 and Grant 61203092, in part by the Fundamental Research Funds for the Central Universities under Grant JUSRP51733B, and in part by Serbian Ministry of Education, Science and Technological Development under Grant 451-03-9/2021-14/200108, and in part by National Science Centre in Poland under Grant 2020/39/B/ST7/01487.
(Corresponding authors: Hongfeng Tao and Vladimir Stojanovic.)

Z. Zhuang and H. Tao are all with the Key Laboratory of Advanced Process Control for Light Industry of Ministry of Education, Jiangnan University, 1800 Lihu Road, Wuxi 214122, China (e-mails: 6191905054@stu.jiangnan.edu.cn; taohongfeng@jiangnan.edu.cn.)

Y. Chen is with the School of Mechanical and Electrical Engineering, Soochow University, 8 Jixue Road, Suzhou, 215137, China (e-mail: yychen90@suda.edu.cn).

V. Stojanovic is with the Faculty of Mechanical and Civil Engineering, Department of Automatic Control, Robotics and Fluid Technique, University of Kragujevac, 36000 Kraljevo, Serbia (e-mail: vladostojanovic@mts.rs).

W. Paszke is with the Institute of Automation, Electronic and Electrical Engineering, University of Zielona Góra, Zielona Góra, Poland (e-mail: w.paszke@issi.uz.zgora.pl).

In practice industrial process, the strict requirement that each trial length must be identical cannot be always satisfied [10]. For instance, when using functional electrical stimulation (FES) to help stroke patients suffering from foot drop to walk, for safety reasons, FES should be applied at least until the initial contact between the foot and the ground was detected in spite of the nonuniform duration on the foot motion [11]. Another example for nonuniform trial lengths is a crane with output constraints [12]. Due to that the crane can not run beyond the neighborhood of a region restricted by some obstacles around, the duration of the tracking is going to be nonidentical when using ILC schemes. Actually, when utilizing ILC methods to perform trajectory tracking tasks in practice, obstacles may occur around the desired trajectory more or less, problems with nonuniform trial lengths thus are indispensable attributed to the output constraints. A further simulation example will be discussed in section VI.

Due to the generality of problems with nonuniform trial lengths, some research on how to relax this restriction has been done in the field of ILC. The main schemes to handle this problem are to do work on the compensation of the missing information that is used for update, which includes research as [13]–[19]. However, different forms of the schemes lead to different control performance. Methods using traditional P-type ILC such as [14], [17], [18], are a kind of lazy pattern, which means lower speed of convergence and poorer robustness to the randomness of nonuniform trial lengths. In [13], [15], [16], [19], the iteration average operator was introduced to improve the utilization of historical information, while the performance of which may become poorer as the number of iteration increases, because the average operator may weaken the effect of the instant learning. Besides, a more effective method, which uses the most recent trial information that still exists at each time instant, was proposed in [20]. Its operation seems like a kind of shelter-from-rain learning and the current time instant only benefits from the most recent existing trial information correspondingly, however, the monotonic convergence can not be ensured. All of the methods mentioned before can not guarantee optimality for a specific trial and thus, the optimal ILC approaches, such as norm optimal ILC [21], [22] and successive projection [23], [24], can be utilized.

Input constraints are also an indispensable part of the industrial process, and should be fully taken into consideration [25], especially when adopting optimal ILC approaches. Some works on optimal ILC design with input constraints have been done, where the ILC problems are usually reformulated to constrained optimization problems for certain trial. A norm optimal ILC for time-varying linear systems with inequality constraints, was revisited and generalized under deterministic, stochastic disturbances and noises in [26]. Also, a successive projection framework for constrained ILC problems was proposed in [27], where the convergence of the optimal ILC algorithms was proved in Hilbert space. Furthermore, data-driven constrained optimal ILC methods for linear and nonlinear systems were proposed in [28] and [29] respectively. However, these constrained optimal ILC methods mentioned above just discuss the feasibility and do not attempt to improve the performance under input constraints.

To better handle optimal ILC problems with input constraints, the interior point methods in numerical optimization theory were introduced. In [30], the barrier method was used to the design of ILC algorithm, where the restriction that global optimal solution of the unconstrained problem should lie in the constraint set was removed. In addition, the ILC design with barrier method was extended to the point-to-point tracking task in [31]. In contrast to barrier method, primal-dual interior point method was used in [32] for ILC design, and the computational complexity of the optimal ILC process was reduced greatly with the sequentially semi-separable structure. Furthermore, note that the interior point designs above are all based on the 2-norm cost function, a modified interior point method was used in optimal ILC design with non-smooth type cost functions in [33]. However, the convergence analysis of all these iterative methods mentioned above is not studied in a theoretical way.

In this paper, an optimal ILC algorithm for problems with nonuniform trial lengths is designed by using the primal-dual interior point method under input constraints. The ILC design problem is reformulated to an constrained optimization problem and thus the primal-dual interior point method can be used to improve each trial's performance of the ILC process. Furthermore, the convergence of the algorithm is analyzed theoretically. The practical implementation of the proposed algorithm is also presented with a further selection of the step length and the initial point. In the end, a mobile robot with two independent driving wheels is chosen to be the numerical simulation example to verify the effectiveness of the proposed algorithm.

The main contributions are summarized as follows:

1) An algorithm for problems with nonuniform trial lengths under input constraints is proposed, which embeds the input constraints into the design so that the constraints on input can be regulated actively.
2) With the primal-dual interior point method design, a lower convergence boundary of the tracking error, than the norm optimal ILC, can be achieved when facing problems with nonuniform trial lengths though the desired input can not be obtained.
3) A framework for convergence analysis of the optimal ILC algorithm with interior point design is presented, which can be extended to serve other convergence analysis of optimal ILC design with input constraints.

This paper is organized as follows. First of all, the problem formulation is addressed in Section II. Section III introduces an ILC algorithm based on a primal-dual interior point method for problems with nonuniform trial lengths under input constraints. The convergence analysis of the ILC algorithm is presented in Section IV. The practical implementation of the algorithm is discussed in Section V. Simulation verification is shown in Section VI. The conclusions are given in Section VII.

The main notations in this paper are listed: $E\{\cdot\}$ and $P\{\cdot\}$ denote the mathematical expectation and the probability of an event, respectively. $\mathbb{N}$ denotes the set of natural number and $\mathbb{R}^n$ and $\mathbb{R}^{n\times m}$ denote the sets of n-dimensional real vectors and $n\times m$ real matrices, respectively. The superscript $(\cdot)$ and T respectively denote the components in a vector the transpose. The inequality notation $\geq$ and $\leq$ for vectors means comparison on each components. $\|\cdot\|_2$ is denoted as $\|\cdot\|$ for simplicity. Other notations will be introduced as needed in the followings.

II. Problem formulation

In this section, system dynamics is introduced with mathematical notations firstly. Then, the modified tracking error of systems with nonuniform trial lengths is formulated under the lifted system framework, together with input constraints, sequentially. Finally, the ILC design problem with nonuniform trial lengths under input constraints is defined.

A. System Dynamics

Consider the following linear time-invariant systems with the state space model form

$$\begin{cases} x_k(t+1) = Ax_k(t) + Bu_k(t), \\ y_k(t) = Cx_k(t), \end{cases} \tag{1}$$

where the subscript $k \in \mathbb{N}$ denotes the trial number and $t \in \mathbb{N}$ stands for the time index. N_d is the desired trial length with $t \in [0,\ N_d]$. $x_k(t) \in \mathbb{R}^n$, $u_k(t) \in \mathbb{R}^l$ and $y_k(t) \in \mathbb{R}^m$ denote the state, input and output vectors, respectively. A, B and C are system matrices with appropriate dimensions with $CB \neq 0$ for controllability of the system. $y_d(t)$ is defined as the desired output trajectory. The initial condition satisfies $E\{x_k(0)\} = x_d(0)$.

Note that the learning or optimization process of ILC is carried out along the trial, so reformulating the system model (1) to a lifted system framework yields

$$y_k = Gu_k + d_k, \tag{2}$$

where

$$u_k = \left[u_k^T(0), u_k^T(1), \ldots, u_k^T(N_d-1)\right]^T, \tag{3}$$

$$y_k = \left[y_k^T(1), y_k^T(2), \ldots, y_k^T(N_d)\right]^T, \tag{4}$$

and G and d_k represent the system model and the effect of the initial conditions respectively, i.e.

$$G = \begin{bmatrix} CB & 0 & 0 & \cdots & 0 \\ CAB & CB & 0 & \cdots & 0 \\ CA^2B & CAB & CB & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ CA^{N_d-1}B & CA^{N_d-2}B & CA^{N_d-3}B & \cdots & CB \end{bmatrix}, \tag{5}$$

$$d_k = \left[\begin{array}{cccc} (CA)^T & (CA^2)^T & \cdots & (CA^{N_d})^T \end{array}\right]^{\mathrm{T}} x_k(0). \quad (6)$$

With $E\{x_k(0)\} = x_d(0)$, denote d_d as

$$d_d = \left[\begin{array}{cccc} (CA)^T & (CA^2)^T & \cdots & (CA^{N_d})^T \end{array}\right]^{\mathrm{T}} x_d(0), \quad (7)$$

as well as the desired output y_d denoted as

$$y_d = \left[y_d^T(1), y_d^T(2), \ldots, y_d^T(N_d)\right]^T. \quad (8)$$

B. Modified ILC Problem Definition

Classical ILC schemes requires that every trial ends at a fixed time of duration. However, when actual trial lengths are not identical to the desired one, although tracking error can not be utilized to update the control signal directly, the error information available is still useful for the following trials to learn. Therefore, to better use such available information for optimal ILC design, we can still append zero signal values to the absent time instances and set the desired trial length as the maximum one. Then, the modified tracking error is defined as

$$e_k = \left[\overbrace{\underbrace{e_k^T(1), \cdots, e_k^T(N_k)}_{N_k}, 0, \cdots, 0}^{N_d}\right]^T, \quad (9)$$

where N_k denotes the actual trial length of the k-th trial. Denote N_m as the minimum actual length, then N_k varies randomly within $\{N_m, N_m + 1, \cdots, N_d\}$, which means there will be $n_s = N_d - N_m + 1$ possible trial lengths in total. Let the probability of the trial length $N_m, N_m + 1, \cdots, N_d$ to be $p_1, p_2, \cdots, p_{n_s}$, then we have

$$\sum_{i=1}^{n_s} p_i = 1, \quad (10)$$

where all $p_i > 0$, for $1 \leq i \leq n_s$. To formulate the modified tracking error with the actual output and the desired one, a random matrix is introduced as

$$M_k = \left[\begin{array}{cc} I_{N_k} \otimes I_m & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{array}\right] \in \mathbb{R}^{(m \cdot N_d) \times (m \cdot N_d)}, \quad (11)$$

where I_l denotes unit matrix with dimension of $l \times l$ and $\mathbf{0}$ denotes zero matrix with appropriate dimension, and the notation $\otimes$ denotes Kronecker product. Then one has

$$e_k = M_k (y_d - y_k). \quad (12)$$

For simplicity, the mathematical expectation of different dimensions in random matrix is seen as same in this paper, which is in line with most practice scenarios. To calculate the mathematical expectation of the random matrix, another random variables $\chi_k(t) \in \{0, 1\}$ is introduced to represent whether the output occurs at time t at the k-th trial as in [13]. Let $\chi_k(t) = 1$ represents the output occurs, and denote its probability as $p(t)$, i.e.

$$p(t) = P\{(\chi_k(t) = 1)\} = \left\{\begin{array}{cc} 1, & t \leq N_m - 1, \\ \sum\limits_{i=t-N_m+1}^{n_s} p_i, & N_m \leq t \leq N_d. \end{array}\right. \quad (13)$$

According to probability theory, we have $E\{\chi_k(t)\} = p(t)$, which gives rise to

$$\begin{aligned} & E\{M_k\} \\ =& diag\left\{\overbrace{1, 1, \cdots, 1,}^{N_m-1} E\{\chi_k(N_m)\}, \cdots, E\{\chi_k(N_d)\}\right\} \otimes I_m \\ =& \, diag\left\{\overbrace{1, 1, \cdots, 1,}^{N_m-1} p(N_m), \cdots, p(N_d)\right\} \otimes I_m \triangleq \bar{M}. \end{aligned} \quad (14)$$

C. Input Constraints

To ensure safety or match the performance of the actuator, input constraints also exist in practice and are often expressed in the form of mathematical inequalities when using ILC. Some practical forms of input constraints in ILC are listed.

- *Input saturation constraint:*

$$u_{\min} \leq u_{k+1} \leq u_{\max}, \quad (15)$$

where $u_{\min}, u_{\max}$ are the upper and lower bounds of the input signal u_{k+1}.

- *Input constraint with respect to the trial index:*

$$\Delta u_{\min} \leq \Delta u_{k+1} = u_{k+1} - u_k \leq \Delta u_{\max}, \quad (16)$$

where $\Delta u_{\min}, \Delta u_{\max}$ are the upper and lower bounds of the rate of input changes with respect to the trial index Δu_{k+1}.

- *Input constraint with respect to the time index:*

$$\delta u_{\min} \leq \delta u_{k+1}(t) = u_{k+1}(t) - u_{k+1}(t-1) \leq \delta u_{\max}, \quad (17)$$

where $\delta u_{\min}, \delta u_{\max}$ are the upper and lower bounds of the rate of input changes with respect to the time index δu_{k+1}.

Denote above three kinds of constraints as one with respect to the input signal u_{k+1} as that in [26]. In other words, the constraint (16) is changed to be

$$\Delta u_{\min} + u_k \leq u_{k+1} \leq \Delta u_{\max} + u_k. \quad (18)$$

Next, assume $\delta u_k(0) = u_k(0)$, then one has

$$\delta u_{k+1} = \mu u_{k+1}, \quad (19)$$

where

$$\mu = \left[\begin{array}{ccccc} I_l & 0_l & \cdots & 0_l & 0_l \\ -I_l & I_l & \cdots & 0_l & 0_l \\ 0_l & -I_l & \ddots & \vdots & 0_l \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0_l & 0_l & \cdots & -I_l & I_l \end{array}\right] \in \mathbb{R}^{(l \cdot N_d) \times (l \cdot N_d)}. \quad (20)$$

Then (17) is changed to be

$$\delta u_{\min} \leq \mu u_{k+1} \leq \delta u_{\max}, \quad (21)$$

which gives rise to

$$\zeta_u u_{k+1} \geq \zeta_{k+1}, \quad (22)$$

where

$$\zeta_u = \begin{bmatrix} \mathrm{I} \\ -\mathrm{I} \\ \mu \\ -\mu \end{bmatrix}, \zeta_{k+1} = \begin{bmatrix} \max\left(u_{\min}, \Delta u_{\min} + u_k\right) \\ -\min\left(u_{\max}, \Delta u_{\max} + u_k\right) \\ \delta u_{\min} \\ -\delta u_{\max} \end{bmatrix}. \tag{23}$$

According to engineering characteristics, the input constraint set $\mathbf{\Omega}$ defined by (22) is usually a convex set.

D. ILC Design Problem

With the definition of the modified ILC problem, the ILC design problem under the input constraints throughout this paper is presented as follows:

Definition 1: The **ILC design problem with nonuniform trial lengths under input constraints** aims at designing an input update law

$$u_{k+1} = f(u_k, u_{k-1}, \cdots, e_k, e_{k-1}, \cdots), \tag{24}$$

which consists of previous trial's input and tracking error under input constraints, such that the modified tracking error converges to zero as $k \to \infty$ in the sense of mathematical expectation along the trials, i.e.

$$\lim_{k\to\infty} \|E\{e_k\}\| = 0. \tag{25}$$

Definition 1 describes the problem to be discussed in this paper with simple and clear mathematical expression, which provides the necessary theoretical basis for the control algorithm design in the following section.

III. Constrained ILC Algorithm Design

In this section, the ILC design problem is reformulated as a constrained optimization problem, which hence can be solved by numerical optimization methods. Moreover, an ILC algorithm combined with a primal-dual interior point method is designed for problem in Definition 1.

A. Reformulated as a Constrained Optimization Problem

When using optimization ideas to design ILC algorithms, a cost function must be defined firstly. For the problems with nonuniform trial lengths in this paper, a cost function with respect to weighted norm of modified error in the sense of mathematical expectation and input increment is defined. As an additional term, the input increment is a necessary condition for optimal ILC algorithms to achieve complete tracking and also results in smoothness. Therefore, the cost function is defined as

$$J\left(E\left\{e_{k+1}\right\}, u_{k+1}\right) = \left\|E\left\{e_{k+1}\right\}\right\|_Q^2 + \left\|u_{k+1} - u_k\right\|_R^2. \tag{26}$$

To obtain the optimization problem, first of all, substitute (12) into the cost function (26) and reformulate it to the form with respect to u_{k+1}, then we have

$$\begin{aligned} J\left(E\left\{e_{k+1}\right\}, u_{k+1}\right) =& u_{k+1}^T \left(G^T \bar{K} G + R\right) u_{k+1} - \\ & 2\left[u_k^T\left(G^T \bar{K} G + R\right) + e_k^T Q \bar{M} G\right] u_{k+1} \\ & + d, \end{aligned} \tag{27}$$

where

$$d = u_k^T R u_k + \left(y_d - d_d\right)^T \bar{K} \left(y_d - d_d\right).$$

When just considering the $(k+1)$th trial, information of the kth trial, as well as trials before it, can be seemed as constants. Therefore, constrained ILC problem is reformulated as quadratic programming problem under inequality constraints as follows:

$$\begin{aligned} \min_{u_{k+1}} \quad & J\left(u_{k+1}\right) = \tfrac{1}{2} u_{k+1}^T H u_{k+1} + c^T u_{k+1} + d \\ & s.t. \quad \zeta_u u_{k+1} - \zeta_{k+1} \geq 0, \end{aligned} \tag{28}$$

where

$$\begin{aligned} H &= 2\left(G^T \bar{K} G + R\right), \\ c^T &= -2\left[u_k^T\left(G^T \bar{K} G + R\right) + e_k^T Q \bar{M} G\right]. \end{aligned}$$

Note that if the weight matrices Q and R are chosen appropriately so that H is a positive definite matrix and the input constraint set Ω is convex, (28) will be a convex constrained optimization problem, whose optimal solution is hence unique.

B. Primal-Dual Interior Point Method Design

Note that interior point methods in numerical optimization are efficient to solve problems with inequality constraints iteratively, and consequently can be adapted to ILC algorithm design. In this paper, primal-dual interior point method is selected to design an algorithm for constrained ILC problems with non-uniform trial lengths. Different from the design using barrier methods in [33], there is only one loop in primal-dual method, where the primal and dual variables are updated simultaneously and thus the convergence speed can be increased in practice.

Define the dual variable $\lambda \in \mathbb{R}^s$ with $s = 4l \cdot N_d$ for symbolic simplification, and note that if H is a positive definite matrix, the optimal solution of problem (28) is unique and satisfies the following Karush-Kuhn-Tucker (KKT) conditions:

$$\begin{aligned} H u_{k+1} + c - \zeta_u^T \lambda &= 0, \\ \zeta_u u_{k+1} - \zeta_{k+1} &\geq 0, \\ \left(\zeta_u u_{k+1} - \zeta_{k+1}\right)^{(i)} \lambda^{(i)} &= 0, \quad i = 1, 2, \ldots, s, \\ \lambda &\geq 0. \end{aligned} \tag{29}$$

Introduce a slack variable $\omega \in \mathbb{R}^s$ to give rise to the modified KKT conditions:

$$\begin{aligned} H u_{k+1} + c - \zeta_u^T \lambda &= 0, \\ \zeta_u u_{k+1} - \zeta_{k+1} - \omega &= 0, \\ \Lambda W \beta &= \varphi \beta, \\ \lambda, \omega &\geq 0, \end{aligned} \tag{30}$$

where

$$\begin{aligned} \Lambda &= diag\left(\lambda^{(1)}, \lambda^{(2)}, \cdots, \lambda^{(s)}\right), \\ W &= diag\left(\omega^{(1)}, \omega^{(2)}, \cdots, \omega^{(s)}\right), \\ \beta &= [1, 1, \cdots, 1]^T \in \mathbb{R}^s, \\ \varphi &= \delta \cdot \theta / s, \end{aligned}$$

with parameter $\delta \in (0, 1)$. $\theta = \lambda^T\omega$ denotes the complementarity in path-following methods, which is also the duality gap for the convex problem (28). For arbitrary $\varphi > 0$, the solution of modified KKT conditions (30) is unique and denote it as $u_{k+1}(\varphi), \lambda(\varphi), \omega(\varphi)$. In addition, $\{u_{k+1}(\varphi), \lambda(\varphi), \omega(\varphi)|\varphi > 0\}$ is the central path of the primal problem (28). By reducing φ continuously, duality gap gets smaller so as to keep approaching to the global optimal solution.

When fixing φ and applying Newton's method, substitute $(u_k + \Delta\bar{u}_{k+1}, \lambda + \Delta\lambda, \omega + \Delta\omega)$ to (30) and ignore the quadratic terms, then we have

$$\begin{bmatrix} -H & \zeta_u^T & 0 \\ \zeta_u & 0 & -I_s \\ 0 & W & \Lambda \end{bmatrix} \begin{bmatrix} \Delta\bar{u}_{k+1} \\ \Delta\lambda \\ \Delta\omega \end{bmatrix} = \begin{bmatrix} \sigma \\ \rho \\ \varphi\beta - \Lambda W\beta \end{bmatrix}, \tag{31}$$

where

$$\begin{aligned} \sigma &= Hu_k + c - \zeta_u^T\lambda, \\ \rho &= \zeta_{k+1} - \zeta_u u_k + \omega, \end{aligned}$$

and $\Delta\bar{u}_{k+1}$, $\Delta\lambda$ and $\Delta\omega$ denote the step directions.

The step directions will be obtained by computing (31), while these step directions is based on the premise that the step length α is 1. However, in path-following methods, the dual and slack variables are ensured to be positive definite as in (30), that is

$$\begin{aligned} \lambda^{(i)} + \alpha\Delta\lambda^{(i)} > 0, \\ \omega^{(j)} + \alpha\Delta\omega^{(j)} > 0, \end{aligned} \quad i, j = 1, 2, \ldots, s. \tag{32}$$

While (32) is the condition that path-following algorithms must meet, a neighborhood of the central path can be defined further to prevent the outputs from coming too close to the boundary, which yields a nontrivial step can be taken along each trial.

Definition 2: Define a neighborhood of the central path of the primal problem (28), which satisfies

$$N(\gamma) = \left\{ (u_{k+1}, \lambda, \omega) \,|\lambda^{(i)}\omega^{(i)} \geq \gamma\theta, i = 1, 2, \ldots, s \right\}, \tag{33}$$

where $\gamma \in (0, 1)$.

Remark 1: According to Definition 2, each pairwise product $\lambda^{(i)}\omega^{(i)} \geq \gamma\theta$ must be at least some small multiple γ of their average value θ. Therefore, reducing γ means encompassing more feasible region.

Remark 2: By introducing the neighborhood (33), there will be a boundary that the step length should stay in. Furthermore, if we choose α_k as the largest value of the boundary in each trial, the algorithm will become more efficient, which also makes up the convergence analysis.

After confirming the step directions and the selection of the step length, an ILC algorithm for problems with nonuniform trial lengths under input constraints based on a primal-dual interior point method is designed as follows:

Algorithm 1: Given system dynamics (1), $\gamma, \delta \in (0, 1)$ and $(u_0, \lambda_0, \omega_0) \in N_{-\infty}(\gamma)$, compute the step directions by (31) and choose α_k as the largest value such that

$$(u_{k+1}, \lambda_{k+1}, \omega_{k+1}) \in N_{-\infty}(\gamma), \tag{34}$$

then an input sequence $\{u_k\}_{k\geq 0}$ for ILC design problem in Definition 1 can be generated by the ILC update law

$$u_{k+1} = u_k + \alpha_k \Delta\bar{u}_{k+1}, \tag{35}$$

together with

$$\lambda_{k+1} = \lambda_k + \alpha_k \Delta\lambda_{k+1}, \tag{36}$$

$$\omega_{k+1} = \omega_k + \alpha_k \Delta\omega_{k+1}. \tag{37}$$

Remark 3: When facing the input constraints, Algorithm 1 can achieve higher convergence accuracy although the desired input can not be obtained because of the primal-dual interior point design. Its performance will be shown in the simulation later.

Remark 4: Different from standard Newton's method, the search direction of each trial in Algorithm 1 is upon the experimental data in the ILC process, instead of calculating the gradient and Hessian matrices.

To explain how Algorithm 1 can solve the problem in Definition 1, a proposition is as follows:

Proposition 1: The input sequence generated by Algorithm 1 iteratively solves the ILC design problem in Definition 1.

Proof: Given initial input signal u_0, with modified problem definition for nonuniform trial lengths before, the input sequence generated by (35) in Algorithm 1 is within input constraints and varies along the trial, and hence can solves the ILC design problem in Definition 1 iteratively. ∎

An algorithm for problems with nonuniform trial lengths under input constraints is designed in this section. In the next section, we will analyze the global convergence of the proposed algorithm.

IV. Convergence Analysis

In this section, the convergence of Algorithm 1 will be analyzed. Before this, an assumption with respect to the input signal is presented as follows:

Assumption 1: There exists a desired input u_d such that the tracking error of systems with nonuniform trial lengths converges to zero in the sense of mathematical expectation.

Assumption 1 ensures that the systems with nonuniform trial lengths are possible to achieve the zero tracking error in the sense of mathematical expectation. While with some input constraints, it would be not possible. Therefore, with input constraint set Ω, $u_d \in \Omega$ is called the perfect tracking being possible and $u_d \notin \Omega$ is called the perfect tracking being not possible.

The analysis of the global convergence consists of two parts. The first part is to find out the global solution of the constrained optimization problem (28), and the second part is to analyze the monotonic and zero convergence property of the tracking error in the sense of mathematical expectation.

In order to find the global solution of (28), an optimal solution that satisfies the conditions (29) should be found out, then the following lemma can be used to show that this solution is global.

Lemma 1: If u_∞ satisfies the conditions (29) for $\lambda^{(i)}$, $i = 1, 2, \ldots, s$, and H is positive definition, then u_∞ is a global solution of (28).

Proof: See Chapter 16 in [34] for more details. ■

With Lemma 1, we are going to show that Algorithm 1 can find out the global solution of (28) as follows:

Theorem 1: When $u_d \in \Omega$ and it means perfect tracking is possible, given systems with nonuniform trial lengths (1), applying Algorithm 1 with given $\delta, \gamma \in (0, 1)$ and positive definite matrix H, then the input sequence $\{u_k\}_{k\geq 0}$ converges to the global solution u_∞ of the constrained optimization problem (28).

Proof: Firstly, analyze the three feasible measurement ρ, σ and θ in duality theory along the trial axis. For the $(k+1)$th trial, we have

$$\rho_{k+1} = \zeta_{k+1} - \zeta_u u_{k+1} + \omega_{k+1}. \tag{38}$$

Substituting (36) and (37) in Algorithm 1 yields

$$\begin{aligned} \rho_{k+1} &= \zeta_{k+1} - \zeta_u u_k + \omega_k - \alpha_k \left(\zeta_u \Delta\bar{u}_{k+1} - \Delta\omega_{k+1}\right) \\ &= \rho_k - \alpha_k \left(\zeta_u \Delta\bar{u}_{k+1} - \Delta\omega_{k+1}\right). \end{aligned} \tag{39}$$

Add the trial index k to (31), then one gives

$$\zeta_u \Delta\bar{u}_{k+1} - \Delta\omega_{k+1} = \rho_k, \tag{40}$$

substituting which to (39) yields

$$\rho_{k+1} = (1 - \alpha_k)\, \rho_k, \tag{41}$$

similarly

$$\sigma_{k+1} = (1 - \alpha_k)\, \sigma_k. \tag{42}$$

For complementarity, there exists

$$\begin{aligned} \theta_{k+1} &= \lambda_{k+1}^T \omega_{k+1} \\ &= \left(\lambda_k + \alpha_k \Delta\lambda_{k+1}\right)^T \left(\omega_k + \alpha_k \Delta\omega_{k+1}\right) \\ &= \theta_k + \alpha_k \left(\Delta\lambda_{k+1}^T \omega_k + \lambda_k^T \Delta\omega_{k+1}\right) \\ &\quad + \alpha_k^2 \cdot \Delta\lambda_{k+1}^T \Delta\omega_{k+1}. \end{aligned} \tag{43}$$

Look also back to (31) with the trial index k, we have

$$\begin{aligned} \Delta\lambda_{k+1}^T \omega_k + \lambda_k^T \Delta\omega_{k+1} &= \beta^T \left(W_k \Delta\lambda_{k+1} + \Lambda_k \Delta\omega_{k+1}\right) \\ &= \beta^T \left(\varphi_k \beta - \Lambda_k W_k \beta\right) \\ &= \delta\theta_k - \theta_k, \end{aligned} \tag{44}$$

substituting which to (43) with $\Delta\lambda^T \Delta\omega = 0$ gives rise to

$$\theta_{k+1} = \left[1 - (1 - \delta)\, \alpha_k\right] \theta_k. \tag{45}$$

Then, we are going to find the boundary that the step length should stay in when (34) is satisfied, which means

$$\lambda_{k+1}^{(i)} \omega_{k+1}^{(i)} \geq \gamma \theta_{k+1}, \quad i = 1, 2, \ldots, s. \tag{46}$$

Expanding the left hand side of (46), one has

$$\begin{aligned} \lambda_{k+1}^{(i)} \omega_{k+1}^{(i)} &= \left(\lambda_k^{(i)} + \alpha_k \Delta\lambda_{k+1}^{(i)}\right)^T \left(\omega_k^{(i)} + \alpha_k \Delta\omega_{k+1}^{(i)}\right) \\ &= \lambda_k^{(i)} \omega_k^{(i)} + \alpha_k \left(\Delta\lambda_{k+1}^{(i)} \omega_k^{(i)} + \lambda_k^{(i)} \Delta\omega_{k+1}^{(i)}\right) \\ &\quad + \alpha_k^2 \Delta\lambda_{k+1}^{(i)} \Delta\omega_{k+1}^{(i)} \\ &\geq \gamma (1 - \alpha_k)\, \theta_k + \alpha_k \delta\theta_k - \alpha_k^2 \left|\Delta\lambda_{k+1}^{(i)} \Delta\omega_{k+1}^{(i)}\right|. \end{aligned} \tag{47}$$

If we find out the boundary of $\Delta\lambda_{k+1}^{(i)} \Delta\omega_{k+1}^{(i)}$ in (47), we can combine (45) and (46) to find the boundary of the step length. Now, we are going to do this.

According (31), we have

$$\Lambda^{-1/2} W^{1/2} \Delta\lambda + \Lambda^{1/2} W^{-1/2} \Delta\omega = (\Lambda W)^{-1/2} \left(\varphi\beta - \Lambda W \beta\right). \tag{48}$$

Note that when applying Newton's method to obtain (31), we ignore the quadratic terms, that is

$$\left(\Lambda^{-1/2} W^{1/2} \Delta\lambda\right)^T \left(\Lambda^{1/2} W^{-1/2} \Delta\omega\right) = \Delta\lambda^T \Delta\omega = 0. \tag{49}$$

Let $v = \Lambda^{-1/2} W^{1/2} \Delta\lambda$ and $q = \Lambda^{1/2} W^{-1/2} \Delta\omega$, which satisfies $v^T q \geq 0$. For $i = 1, 2, \ldots, s$, define $F_1 = \left\{i | v^{(i)} q^{(i)} \geq 0\right\}$ and $F_2 = \left\{i | v^{(i)} q^{(i)} < 0\right\}$, and define $\Delta\Lambda$ and ΔW as

$$\begin{aligned} \Delta\Lambda &= diag\left(\Delta\lambda^{(1)}, \Delta\lambda^{(2)}, \cdots, \Delta\lambda^{(s)}\right), \\ \Delta W &= diag\left(\Delta\omega^{(1)}, \Delta\omega^{(2)}, \cdots, \Delta\omega^{(s)}\right). \end{aligned}$$

Since we know that $\left|\Delta\lambda_{k+1}^{(i)} \Delta\omega_{k+1}^{(i)}\right| \leq \|\Delta\Lambda\Delta W \beta\|$, it is necessary to show that

$$\begin{aligned} \|\Delta\Lambda\Delta W\beta\| &= \left\|\left(\Lambda^{-1/2} W^{1/2} \Delta\Lambda\right)\left(\Lambda^{1/2} W^{-1/2} \Delta W\right)\beta\right\| \\ &= \left(\left\|\left[v^{(i)} q^{(i)}\right]_{i\in F_1}\right\|^2 + \left\|\left[v^{(i)} q^{(i)}\right]_{i\in F_2}\right\|^2\right)^{1/2} \\ &\leq \left(2\left\|\left[v^{(i)} q^{(i)}\right]_{i\in F_1}\right\|_1^2\right)^{1/2} \\ &\leq \sqrt{2}\left\|\left[\frac{1}{4}\left(v^{(i)} + q^{(i)}\right)^2\right]_{i\in F_1}\right\|_1 \\ &= 2^{-3/2} \sum_{i\in F_1} \left(v^{(i)} + q^{(i)}\right)^2 \\ &\leq 2^{-3/2} \sum_{i=1}^{m} \left(v^{(i)} + q^{(i)}\right)^2 \leq 2^{-3/2} \|v + q\|^2, \\ &= 2^{-3/2} \left\|\Lambda^{-1/2} W^{1/2} \Delta\lambda + \Lambda^{1/2} W^{-1/2} \Delta\omega\right\|^2, \end{aligned} \tag{50}$$

with some mathematical definitions and transformations before. Substitute (48) to (50), then we have

$$\|\Delta\Lambda\Delta W\beta\| \leq 2^{-3/2} \left\|(\Lambda W)^{-1/2} \left(\varphi\beta - \Lambda W\beta\right)\right\|^2. \tag{51}$$

Expanding the squared Euclidean norm with $\lambda^T \omega = \theta$, $\beta^T \beta =$

s and $\lambda^{(i)}\omega^{(i)} \geq \gamma\theta$ in (33) gives rise to

$$\begin{aligned}\left|\Delta\lambda_{k+1}^{(i)}\Delta\omega_{k+1}^{(i)}\right| &\leq \|\Delta\Lambda\Delta W\beta\| \\ &\leq \begin{array}{c} 2^{-3/2}\left[\beta^T\Lambda W\beta - 2\varphi\beta^T\beta\right. \\ \left.+(\varphi\beta)^T(\Lambda W)^{-1}\varphi\beta\right]\end{array} \\ &= 2^{-3/2}\left[\theta - 2\delta\theta + \varphi^2\sum_{i=1}^{s}\frac{1}{\lambda^{(i)}\omega^{(i)}}\right] \\ &\leq 2^{-3/2}\left[\theta - 2\delta\theta + \varphi^2\frac{s}{\gamma\theta}\right] \\ &= 2^{-3/2}\left[\theta - 2\delta\theta + \frac{\delta^2}{\gamma\cdot s}\theta\right] \\ &\leq 2^{-3/2}\left(1+\frac{1}{\gamma}\right)\theta,\end{aligned} \tag{52}$$

for $0<\delta<1$, which eventually finds the boundary.

Combining (47) with (52), if (46) is going to be met, the followings should be satisfied

$$\begin{aligned}\gamma(1-\alpha_k)\theta_k + \alpha_k\delta\theta_k - \alpha_k^2 2^{-3/2}\left(1+\frac{1}{\gamma}\right)\theta_k \\ \geq \gamma\left[1-(1-\delta)\alpha_k\right]\theta_k,\end{aligned} \tag{53}$$

that is

$$0 < \alpha_k \leq 2^{3/2}\delta\gamma\frac{1-\gamma}{1+\gamma}. \tag{54}$$

According to Algorithm 1, the step length α_k is at least as long as the upper bound of (54), so we have

$$\alpha_k \geq 2^{3/2}\delta\gamma\frac{1-\gamma}{1+\gamma} = z. \tag{55}$$

After achieving (55), we finally find the value away that the step lengths remain bigger than and the value is away from zero. So From (41) and the bound on α_k, it follows that

$$\|\rho_{k+1}\|_1 \leq (1-z)\|\rho_k\|_1 \leq \cdots \leq (1-z)^{k+1}\|\rho_0\|_1, \tag{56}$$

and similarly for (42), it follows that

$$\|\sigma_{k+1}\|_1 \leq (1-z)\|\sigma_k\|_1 \leq \cdots \leq (1-z)^{k+1}\|\sigma_0\|_1. \tag{57}$$

For (45), it follows that

$$\theta_{k+1} \leq \left[1-(1-\delta)z\right]\theta_k \leq \ldots \leq \left[1-(1-\delta)z\right]^{k+1}\theta_0. \tag{58}$$

Combining the results of (56), (57) and (58) with $\gamma,\delta \in (0,1)$, a solution u_∞ that satisfies the KKT condition (29) is eventually found as $k \to \infty$. Furthermore, according to Lemma 1, due to H is positive definite, u_∞ is the global solution of the constrained optimization problem (28) and the proof is completed. ■

Remark 5: Different with (56) and (57), the duality gap θ in (58) goes down by a smaller factor $1-(1-\delta)z$. Substitute z into it to get the convergence factor of the duality gap, which is represented as $1-2^{3/2}\gamma\frac{1-\gamma}{1+\gamma}\cdot\delta(1-\delta)$. By choosing the appropriate δ, the convergence speed to the global optimal solution can become faster.

After finishing finding the global solution u_∞ of the constrained optimization problem (28) as the first part, the relationship between u_∞ and u_d in Assumption 1 is explored to show the convergence property of Algorithm 1.

Theorem 2: When $u_d \in \Omega$ and it means perfect tracking is possible, given systems with nonuniform trial lengths (1), applying Algorithm 1 with appropriate parameters yields

$$\|E\{e_k\}\| \geq \|E\{e_{k+1}\}\|, \tag{59}$$

together with

$$\lim_{k\to\infty}\|E\{e_{k+1}\}\| = 0. \tag{60}$$

Proof: Considering the kth cost function, when applying Algorithm 1, the $(k+1)$th input always gives rise to

$$\begin{aligned}J(E\{e_k\},u_k) &= \|E\{e_k\}\|_Q^2 \\ &\geq J(E\{e_{k+1}\},u_{k+1}) \\ &= \|E\{e_{k+1}\}\|_Q^2 + \|u_{k+1}-u_k\|_R^2,\end{aligned} \tag{61}$$

which yields the monotonic convergence (59).

In addition, substitute (35) to (12) and fix $\Delta u_{k+1}=0$, then we have

$$E\{e_{k+1}\} = E\{e_k\} - \bar{M}G\alpha_k\Delta\bar{u}_{k+1} = E\{e_k\}, \tag{62}$$

which means $(E\{e_k\},u_{k+1})$ is also a feasible point in convex set Ω. Then we have

$$\begin{aligned}J(E\{e_k\},u_k) - \Delta u_{k+1}^T R\Delta u_{k+1} &= J(E\{e_k\},u_{k+1}) \\ &\geq J(E\{e_{k+1}\},u_{k+1}) \geq 0,\end{aligned} \tag{63}$$

which gives rise to

$$J(E\{e_0\},u_0) \geq J(E\{e_{k+1}\},u_{k+1}) + \sum_{i=1}^{k+1}\Delta u_i^T R\Delta u_i \geq 0. \tag{64}$$

Due to the fact that $J(E\{e_0\},u_0) < \infty$, we have

$$\Delta u_\infty = \lim_{k\to\infty}\Delta u_{k+1} = 0. \tag{65}$$

Recall that $(E\{e_\infty\},u_\infty)$ is the global solution of the constrained optimization problem (28), so any direction of the directional derivative with respect to the cost function (26) at the point $(E\{e_\infty\},u_\infty)$ is no less than zero. Considering the directional derivative from $(E\{e_\infty\},u_\infty)$ to $(0,u_d)$ in Assumption 1, it follows that

$$\begin{aligned}&\nabla J^T|_{(E\{e_\infty\},u_\infty)}\cdot\begin{bmatrix}-E\{e_\infty\}\\ u_d-u_\infty\end{bmatrix} \\ &= \begin{bmatrix}E\{e_\infty^T\}Q & \Delta u_\infty^T R\end{bmatrix}\cdot\begin{bmatrix}-E\{e_\infty\}\\ u_d-u_\infty\end{bmatrix} \\ &= -E\{e_\infty^T\}QE\{e_\infty\} + \Delta u_{k+1}^T R(u_d-u_\infty) \geq 0,\end{aligned} \tag{66}$$

which gives rise to

$$\Delta u_\infty^T Ru_d \geq E\{e_\infty^T\}QE\{e_\infty\} + \Delta u_{k+1}^T Ru_\infty \geq 0, \tag{67}$$

which yields $\|E\{e_\infty\}\| = 0$, and zero convergence property (60) is finally obtained. ■

Remark 6: Theorem 2 reveals that when choosing appropriate Q and R, the tracking error of Algorithm 1 also converges to zero in the sense of mathematical expectation with $u_\infty = u_d$. That is, the sequence $\{u_k\}_{k\geq 0}$ generated by Algorithm 1 can also converge to the desired input in

Assumption 1.

After finishing the analysis of the global convergence with two parts above, a corollary is deduced when perfect tracking is not possible.

For convex quadratic programming problem under inequality constraints, the global solution must be on the the boundary of the convex constraint set Ω. When perfect tracking is not possible, denote the the global optimal solution under inequality constraints as u_s^*, then each components of u_s^* may reach to the boundary or partially, which depends on the relationship between Ω and u_d. Moreover, a corollary is given rise to

Corollary 1: When $u_d \notin \Omega$ and it means perfect tracking is not possible, given systems with nonuniform trial lengths (1), applying Algorithm 1 with appropriate parameters, it follows that

$$\lim_{k\to\infty} \|E\{e_k\}\| = \left\|\bar{M}\left(y_d - Gu_s^* - d_d\right)\right\|. \tag{68}$$

Proof: With input constraints, the input signal can not converge to u_d, which yields

$$\lim_{k\to\infty} u_k = u_s^*. \tag{69}$$

Then, according to the result of Theorem 2, (68) is obtained. ■

The convergence boundary is obtained in Corollary 1 when the perfect tracking is not possible. Furthermore, with the neighborhood defined in Definition 2, Algorithm 1 can take a nontrivial step along each trial so as to get closer to the theoretical boundary derived from Corollary 1, which is verified in the simulation part by comparing with the norm optimal ILC method.

V. Practical Implementation of the Algorithm

In this section, a practical implementation of Algorithm 1 is provided by confirming the selection of the step length and the choice of the initial point. Finally, the practical algorithm is presented.

A. Selection of Step Length and Initial Point

Actually, the condition for the step length (34) in Algorithm 1 is utilized to reveal a property that the step length remains bounded away from zero as in Theorem 1. While in practical implementation, a further selection of the step length, which can achieve faster convergence in the iterative process, should be discussed.

Recall that the dual and slack variables should be positive definite so that (32) is satisfied. Therefore, different step lengths are indeed generated

$$\begin{aligned} \lambda^{(i)} + \alpha^{pri}\Delta\lambda^{(i)} > 0, \\ \omega^{(j)} + \alpha^{dual}\Delta\omega^{(j)} > 0, \end{aligned} \qquad i,j = 1,2,\ldots,s, \tag{70}$$

where α^{pri} and α^{dual} denote the step lengths that ensure the positive definite of the dual and slack variables respectively. Introduce a parameter $\tau \in (0,1)$ to achieve the equality, then we have

$$\alpha^{pri} = \max\left\{\alpha : \tau\lambda^{(i)} + \alpha\Delta\lambda^{(i)} \geq 0, i = 1,2,\ldots,s\right\}, \tag{71}$$

$$\alpha^{dual} = \max\left\{\alpha : \tau\omega^{(j)} + \alpha\Delta\omega^{(j)} \geq 0, j = 1,2,\ldots,s\right\}, \tag{72}$$

where $\alpha \in (0,1]$ and τ is often close to but strictly less than 1 so as to accelerate the convergence.

As to the initial point, positive definite of λ and ω is also essential. Moreover, it is better to choose an initial point that is far away from the boundary of the region $\lambda, \omega \geq 0$. In this case, Algorithm 1 may take long steps in the first few trials. More details about the choice of the initial point are presented in [35].

B. A Practical Algorithm

After confirming the practical step length selection as well as the instruction of the initial point, a practical algorithm is presented as follows:

Algorithm 2 Practical Implementation of Algorithm 1

Input: System dynamics (1), weight matrices $Q, R \geq 0$, u_0, λ_0, ω_0 with $u_0 \in \Omega$ and $\lambda_0, \omega_0 \geq 0$, parameters $\delta, \gamma \in (0,1)$ and $\tau \in (0,1)$.

Output: A sequence $\{u_k\}_{k\geq 0}$ to solve the ILC design problem in Definition 1.

1: **for** $k = 0,1,2,\cdots$ **do**
2: Calculate the two step lengths α_k^{pri} and α_k^{dual} using (71) and (72);
3: Set the step length $\alpha_k = \min\left\{\alpha_k^{pri}, \alpha_k^{dual}\right\}$ with $\alpha_k \in \left[2^{3/2}\delta\gamma\frac{1-\gamma}{1+\gamma}, 1\right]$;
4: Calculate σ, ρ and φ in (31);
5: Solve (31) for the step directions $\Delta\bar{u}_{k+1}$, $\Delta\lambda_{k+1}$ and $\Delta\omega_{k+1}$;
6: Set

$$\begin{aligned} u_{k+1} &\leftarrow u_k + \alpha_k\Delta\bar{u}_{k+1}, \\ \lambda_{k+1} &\leftarrow \lambda_k + \alpha_k\Delta\lambda_{k+1}, \\ \omega_{k+1} &\leftarrow \omega_k + \alpha_k\Delta\omega_{k+1}; \end{aligned}$$

7: **end for**
8: **return** $\{u_k\}_{k\geq 0}$.

With this practical algorithm, the ILC design problem in Definition 1 can be solved, which will be verified in the next section on a numerical simulation example.

VI. Numerical Simulation Verifications

To verify the effectiveness of the proposed algorithm in this paper, a numerical model of mobile robot with two independent driving wheels as in [36] is employed. By controlling the driving voltages u_r and u_l of each wheel, the linear velocity v and the azimuth ϕ of the mobile robot can be taken in control so that the mobile robot can perform trajectory tracking tasks on a fixed two dimensional rectangular coordinate system.

A. Simulation Specification

Define the state variable of the mobile robot as $x = \begin{bmatrix} v & \phi & \dot{\phi} \end{bmatrix}^T$, the input variable as $u = [u_r \quad u_l]^T$, and the output variable as $[v \quad \phi]^T$. Set the repeat operation period as $T = 2s$ and the sampling period as $0.05s$, which yields $N_d = 40$. Then, the discrete-time state-space parameters are

$$A = \begin{bmatrix} 0.9975 & 0 & 0 \\ 0 & 1 & 0.0499 \\ 0 & 0 & 0.9955 \end{bmatrix}, B = \begin{bmatrix} 0.0125 & 0.0125 \\ -0.0021 & -0.0042 \\ -0.0833 & -0.1666 \end{bmatrix}, \\ C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}. \tag{73}$$

When a mobile robot is going to move along a specific desired trajectory under control of ILC algorithms, some output constraints usually arise from the obstacles on the trajectory, which may lead to the problem with nonuniform trial lengths. In fact, the running trajectory of the robot is always larger deviation from the desired trajectory in the previous few trials, and get closer as the progress of the ILC process. Although the obstacles are sometimes far from the desired trajectory, the output is possible to be constrained, and thus the situation that trial lengths vary still happens.

In this paper, the actual trial length N_k is set to satisfy the discrete uniform distribution, which means N_k vary randomly between an integer set. Let the set be $\{33, 34, \cdots, 40\}$ and thus $p_i = 1/8$. It should be pointed out that the algorithm proposed in this paper can be used as long as the probability distribution of the actual trial length can be known. Set the initial state and the initial input signal as $x_d(0) = [0, 0, 0]^T$ and $u_0(t) = 0, 0 \le t \le N_d - 1$, respectively. Set $k = 30$ and $N_{30} = N_d$ for better observation.

Furthermore, note that the mobile robot system (73) is a linear coupling MIMO system and can be decoupled

$$\begin{bmatrix} u_r \\ u_l \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}. \tag{74}$$

Then, the linear velocity v is controlled by u_1 alone and the azimuth ϕ is controlled by both u_1 and u_2. The control procedure in this simulation is to control the linear velocity v firstly and then let the u_1 be a disturbance so that the azimuth ϕ can be control by u_2 alone. In this way, the robustness of the proposed algorithm to disturbance can also be verified. Set the desired trajectory of the linear velocity and the azimuth be $\nu_d = 1\ m/s$ and $\phi_d = \pi t\ rad$ respectively, then the desired trajectory of the mobile robot is round.

B. Performance of the Proposed Algorithm

When applying the ILC algorithm proposed in this paper, some other parameters should be confirmed. Choose $\delta = 0.1$ and initial the dual variable and slack variable such that $\lambda_0 = 2I_{s\times1}$ and $\omega_0 = 2I_{s\times1}$. Choose weight matrices $Q = I$ and $R = 0.001I$, which also ensures the positive definite of matrix H. When perfect tracking is possible, set constraints on u_1 being $u_{\max}, u_{\min} = \pm150V$, $\Delta u_{\max}, \Delta u_{\min} = \pm100V$ and $\delta u_{\max}, \delta u_{\min} = \pm100V$, and constraints on u_2 being $u_{\max}, u_{\min} = \pm20V$, $\Delta u_{\max}, \Delta u_{\min} = \pm20V$ and

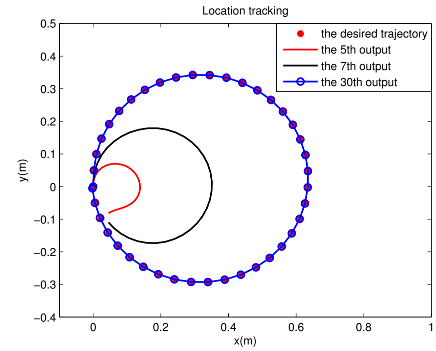

Fig. 1. Location tracking of the $5th$, $7th$ and $30th$ trials when the perfect tracking is possible.

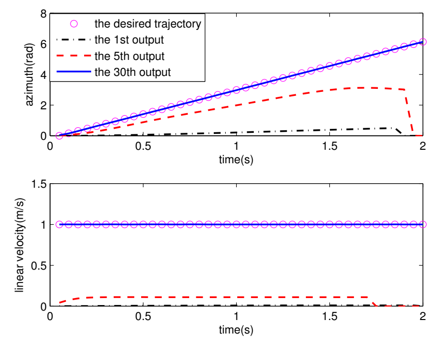

Fig. 2. Output of linear velocity and azimuth for the $1st$, $5th$ and $30th$ trials when the perfect tracking is possible.

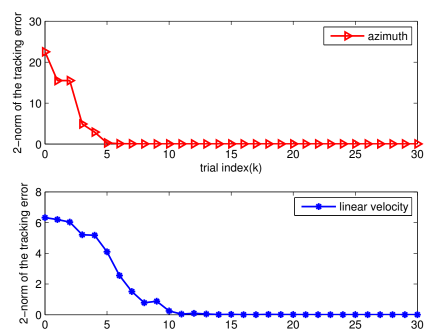

Fig. 3. Tracking error of linear velocity and azimuth along the trial axis when the perfect tracking is possible.

$\delta u_{\max}, \delta u_{\min} = \pm20V$. Recall that the proposed algorithm embeds the input constraints into the ILC process actively, which means output range as well as performance can be regulated independently. It should be also pointed that the real input constraints on u_r and u_l can also be transformed into the constraints on u_1 and u_2 by (74), and so we directly do it on u_1 and u_2 for simplification.

Simulation results are shown in Fig. 1 to Fig. 4. When the perfect tracking is possible, the tracking trajectories of the $5th$, $7th$ and $30th$ trials are shown in Fig. 1. The $5th$ and the $7th$ trials do not run a complete trajectory because of the situation that trial lengths vary as explained before. Fig. 2 shows the

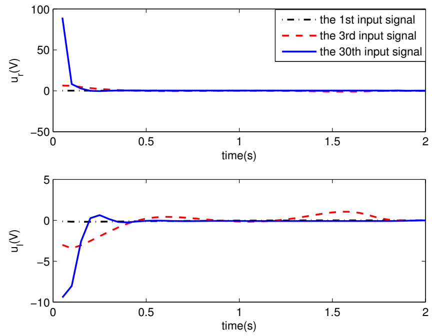

Fig. 4. Input signals of the right and left wheels for the $1st$, $3rd$ and $30th$ trials when the perfect tracking is possible.

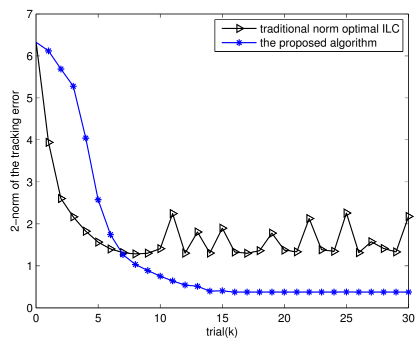

Fig. 5. Tracking error of linear velocity when $u_{\max}, u_{\min} = \pm 50V$, $\Delta u_{\max}, \Delta u_{\min} = \pm 10V$ and $\delta u_{\max}, \delta u_{\min} = \pm 100V$.

output of the linear velocity and the azimuth, whose tracking errors along the trial axis are shown in Fig. 3. The input signals of the right and left wheels for the $1st$, $3rd$ and $30th$ trials are shown in Fig. 4, all of which are under the input constraints set before.

When the perfect tracking is not possible, to illustrate the advantages of the proposed algorithm, a comparative example is presented. Choose the norm optimal ILC law in [22] as a comparison, and choose the weight matrices $Q = I$ and $R = 0.001I$ too. Set the saturation constraints for u_1 as $u_{\max}, u_{\min} = \pm 50V$ and the input constraints with respect to the trial index as $\Delta u_{\max}, \Delta u_{\min} = \pm 12V$, and keep $\delta u_{\max}, \delta u_{\min}$ unchanged. Then the tracking error of the linear velocity is as shown in Fig. 5. Though the convergence speed of the proposed algorithm is a litter lower at the first few trials, a lower convergence boundary as well as monotonic convergence property are obtained, which also verifies the Corollary 1. To further show the performance of the proposed algorithm with input constraint along the time axis, we just change $\delta u_{\max}, \delta u_{\min}$ from $\pm 100V$ to $\pm 12V$, which is shown in Fig. 6. The proposed algorithm can still keep its better convergence than the norm optimal ILC.

VII. Conclusion and Future Work

This paper proposed an optimal ILC algorithm for linear time-invariant MIMO systems with nonuniform trial lengths under input constraints. By reformulating the optimal ILC

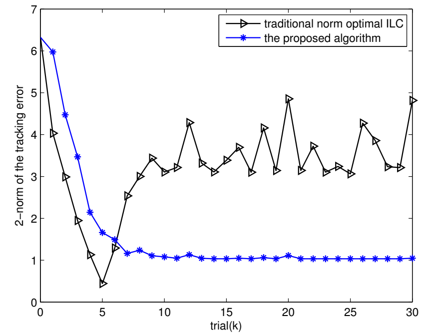

Fig. 6. Tracking error of linear velocity when $u_{\max}, u_{\min} = \pm 50V$, $\Delta u_{\max}, \Delta u_{\min} = \pm 10V$ and $\delta u_{\max}, \delta u_{\min} = \pm 12V$.

problem with input constraints into an constrained optimization problem, the primal-dual interior point method was used to handle it, and thus the proposed algorithm can achieve a better convergence performance when the perfect tracking is not possible. Moreover, with the modified cost function, the convergence of the proposed algorithm for problems with nonuniform trial lengths was analyzed theoretically, and a corollary was achieved under the circumstance that the perfect tracking is not possible. The effectiveness of the proposed algorithm was verified on a mobile robot model by comparing with the norm optimal ILC under the same input constraints.

The future work includes experimental verification to research on the practice performance of the proposed algorithm. Also, the proof of the robustness to modeling uncertainty and non-repeatable disturbance is going to be addressed in the future.

References

[1] S. Hao, T. Liu, and E. Rogers, “Extended state observer based indirect-type ILC for single-input single-output batch processes with time- and batch-varying uncertainties,” *Automatica*, vol. 112, p. 108673, 2020.

[2] H. Tao, W. Paszke, E. Rogers, H. Yang, and K. Galkowski, “Iterative learning fault-tolerant control for differential time-delay batch processes in finite frequency domains,” *J. Process Control*, vol. 56, pp. 112–128, Aug. 2017.

[3] X. Jin, “Fault-tolerant iterative learning control for mobile robots non-repetitive trajectory tracking with output constraints,” *Automatica*, vol. 94, pp. 63–71, 2018.

[4] C. T. Freeman, *Control System Design for Electrical Stimulation in Upper Limb Rehabilitation*. Cham, Switzerland: Springer, 2016.

[5] S. Sa-e, C. T. Freeman, and K. Yang, “Iterative learning control of functional electrical stimulation in the presence of voluntary user effort,” *Control Eng. Pract.*, vol. 96, p. 104303, Mar. 2020.

[6] S. Arimoto, S. Kawamura, and F. Miyazaki, “Bettering operation of robots by learning,” *J. Robot. Syst.*, vol. 1, no. 2, pp. 123–140, 1984.

[7] D. A. Bristow, M. Tharayil, and A. G. Alleyne, “A survey of iterative learning control : A learning-based method for high-performance tracking control,” *IEEE Control Syst. Mag.*, vol. 26, no. 3, pp. 96–114, Jun. 2006.

[8] H.-S. Ahn, Y. Chen, and K. L. Moore, “Iterative learning control: Brief survey and categorization,” *IEEE Trans. Syst., Man, Cybern. C*, vol. 37, no. 6, pp. 1099–1121, Nov. 2007.

[9] D. Shen and Y. Wang, “Survey on stochastic iterative learning control,” *J. Process Control*, vol. 24, no. 12, pp. 64–77, 2014.

[10] D. Shen and X. Li, “A survey on iterative learning control with randomly varying trial lengths: Model, synthesis, and convergence analysis,” *Annu. Rev. Control*, vol. 48, pp. 89–102, 2019.

[11] T. Seel, C. Werner, J. Raisch, and T. Schauer, "Iterative learning control of a drop foot neuroprosthesis - generating physiological foot motion in paretic gait by automatic feedback control," *Control Eng. Pract.*, vol. 48, pp. 87–97, 2016.

[12] M. Guth, T. Seel, and J. Raisch, "Iterative learning control with variable pass length applied to trajectory tracking on a crane with output constraints," in *Proc. 52nd IEEE Conf. Decis. Control*, Dec. 2013, pp. 6676–6681.

[13] X. Li, J.-X. Xu, and D. Huang, "An iterative learning control approach for linear systems with randomly varying trial lengths," *IEEE Trans. Autom. Control*, vol. 59, no. 7, pp. 1954–1960, Jul. 2014.

[14] D. Shen, W. Zhang, and Y. Wang, "On almost sure and mean square convergence of p-type ILC under randomly varying iteration lengths," *Automatica*, vol. 63, pp. 359–365, 2016.

[15] X. Li, J.-X. Xu, and D. Huang, "Iterative learning control for nonlinear dynamic systems with randomly varying trial lengths," *Int. J. Adapt. Control Signal Process.*, vol. 29, no. 11, pp. 1341–1353, 2015.

[16] X. Li and D. Shen, "Two novel iterative learning control schemes for systems with randomly varying trial lengths," *Syst. Control Lett.*, vol. 107, pp. 9–16, 2017.

[17] T. Seel, T. Schauer, and J. Raisch, "Monotonic convergence of iterative learning control systems with variable pass length," *Int. J. Control*, vol. 90, no. 3, pp. 393–406, 2017.

[18] D. Meng and J. Zhang, "Deterministic convergence for learning control systems over iteration-dependent tracking intervals," *IEEE Trans. Neural Netw. Learn. Syst.*, vol. 29, no. 8, pp. 3885–3892, Aug. 2018.

[19] J. Shi, J. Xu, J. Sun, and Y. Yang, "Iterative learning control for time-varying systems subject to variable pass lengths: Application to robot manipulators," *IEEE Trans. Ind. Electron.*, vol. 67, no. 10, pp. 8629–8637, Oct. 2020.

[20] X. Jin, "Iterative learning control for MIMO nonlinear systems with iteration-varying trial lengths using modified composite energy function analysis," *IEEE Trans. Cybern.*, pp. 1–11, Feb. 2020.

[21] N. Amann, D. H. Owens, and E. Rogers, "Iterative learning control using optimal feedback and feedforward actions," *Int. J. Control*, vol. 65, no. 2, pp. 277–293, 1996.

[22] K. L. Barton and A. G. Alleyne, "A norm optimal approach to time-varying ilc with application to a multi-axis robotic testbed," *IEEE Trans. Control Syst. Technol.*, vol. 19, no. 1, pp. 166–180, Jan. 2011.

[23] B. Chu and D. H. Owens, "Accelerated norm-optimal iterative learning control algorithms using successive projection," *Int. J. Control*, vol. 82, no. 8, pp. 1469–1484, 2009.

[24] Y. Chen, B. Chu, and C. T. Freeman, "Generalized iterative learning control using successive projection: Algorithm, Convergence, and Experimental Verification," *IEEE Trans. Control Syst. Technol.*, vol. 28, no. 6, pp. 2079–2091, Nov. 2020.

[25] S. Tarbouriech, G. Garcia, and A. H. Glattfelder, *Advanced Strategies in Control Systems with Input and Output Constraints*. Springer, 2007.

[26] J. H. Lee, K. S. Lee, and W. C. Kim, "Model-based iterative learning control with a quadratic criterion for time-varying linear systems," *Int. J. Control*, vol. 36, no. 5, pp. 641–657, 2000.

[27] B. Chu and D. H. Owens, "Iterative learning control for constrained linear systems," *Int. J. Control*, vol. 83, no. 7, pp. 1397–1413, 2010.

[28] P. Janssens, G. Pipeleers, and J. Swevers, "A data-driven constrained norm-optimal iterative learning control framework for LTI systems," *IEEE Trans. Control Syst. Technol.*, vol. 21, no. 2, pp. 546–551, Mar. 2013.

[29] R. Chi, X. Liu, R. Zhang, Z. Hou, and B. Huang, "Constrained data-driven optimal iterative learning control," *J. Process Control*, vol. 55, pp. 10–29, 2017.

[30] S. Mishra, U. Topcu, and M. Tomizuka, "Optimization-based constrained iterative learning control," *IEEE Trans. Control Syst. Technol.*, vol. 19, no. 6, pp. 1613–1621, Nov. 2011.

[31] C. T. Freeman and Y. Tan, "Iterative learning control with mixed constraints for point-to-point tracking," *IEEE Trans. Control Syst. Technol.*, vol. 21, no. 3, pp. 604–616, May 2013.

[32] A. Haber, R. Fraanje, and M. Verhaegen, "Linear computational complexity design of constrained optimal ILC," in *Proc. 50th IEEE Conf. Decision and Control and European Control Conf.*, Orlando, FL, USA, 2011, pp. 5343–5348.

[33] Y. Guo and S. Mishra, "Constrained optimal iterative learning control with mixed-norm cost functions," *Mechatronics*, vol. 43, pp. 56–65, 2017.

[34] J. Nocedal and S. J. Wright, *Numerical Optimization*, 2nd ed. New York, NY: Springer, 2006.

[35] E. A. Yildirim and S. J. Wright, "Warm-start strategies in interior-point methods for linear programming," *SIAM J. Optim.*, vol. 12, no. 3, pp. 782–810, 2002.

[36] K. Watanabe, J. Tang, M. Nakamura, S. Koga, and T. Fukuda, "A fuzzy-gaussian neural network and its application to mobile robot control," *IEEE Trans. Control Syst. Technol.*, vol. 4, no. 2, pp. 193–199, Mar. 1996.