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FOURTH ORDER COMPUTATIONAL SPLINE METHOD FOR TWO-PARAMETER SINGULARLY PERTURBED BOUNDARY VALUE PROBLEM

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The current research work considers a two-parameter singularly perturbed two-point boundary value problem. Here, we suggest a computational scheme derived by using an exponential spline for the numerical solution of the problem on a uniform mesh. The proposed numerical scheme is analyzed for convergence and an accuracy of $O(h^4)$ is achieved. Numerical experiments are considered to validate the efficiency of the spline method, and compared comparison with the existing method to prove the superiority of the proposed scheme.

Key words: spline, two-parameter singularly perturbed two-point boundary value problem, dual layer, characteristic equation, convergence.

1. Introduction

Numerous physical problems are often connected with the solution of boundary value problems with several small parameters. Problems similar to these come up frequently in the research on transport processes in chemistry and biology [1], the theory of lubrication [2] and in the theory of chemical reactors [3]. O'Malley [4, 5] investigated the asymptotic performance and qualitative features of the solution to the steady-state version of two-parameter problems. He found that the ratio of convection coefficient μ^2 to the diffusion coefficient ε determines the limiting behaviour of the asymptotic solution of the two-parameter singularly perturbed boundary value problem [TPSPBVP] of order two. Therefore, the ratio of μ^2 to ε is crucial in understanding the asymptotic behaviour of these issues.

It is widely known that the analytical solution of the singularly perturbed boundary value problem approaches a discontinuous limit and the appearance of boundary or interior layers as the perturbation parameter approaches zero. This kind of problem has solutions that are layered, meaning that there are thin areas in the domain of the differential equation where the derivatives of the solutions are quite large. Due to the existence of boundary and/or inner layers, numerically solving singularly perturbed differential equations presents significant computing challenges. Among the many articles and books that describe several approaches to solving singly perturbed problems, we highlight those by Bender and Orszag [6], Doolan *et al.* [7] and Miller *et al.* [8].

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A defect correction technique was devised by Kadalbajoo and Jha [9] for numerically solving the TPSPBVP on the Bakhvalov-Shishkin mesh. A model of the linear convection diffusion reaction problem was studied by Lin and Roos [10]. The authors used the method of barrier functions to derive sharp bounds on the solution's derivatives. In order to solve a class of TPSPBVP, the authors of [11] introduced the B-spline collocation method on a piecewise uniform Shishkin mesh. Streamline-diffusion FEM was first introduced by Torsten Linss in [12] to address a TPSPBVP of reaction-convection-diffusion type that gives rise to two boundary layers. Gracia *et al.* [13] devised a monotone numerical method for solving a TPSPBVP. A piecewise uniform Shishkin mesh is combined with the monotone operator in this method. Theoretically, it has been demonstrated that the error constants of the asymptotic error bound in the maximum norm are independent of the perturbation parameters.

To solve a TPSPBVP of a semilinear equation, the authors in [14] suggested a uniformly convergent numerical approach involving an exponential spline on a Shishkin mesh. An approximation method was developed by Kumar *et al.* [15] for solving a TPSPBVP with layers at both ends. The authors of this technique conceptually split the area into an interior and exterior section. A zeroth order asymptotic expansion is utilised to estimate the solution in the outside region, while the B-spline collocation approach is used in the inner region.

This paper proposes a higher-order numerical method for addressing a TPSPBVP. In Section 2, we describe the problem's statement and its characteristics. The spline methodology is outlined in Section 3. In Section 4, the author describes a numerical technique that utilizes splines to address the problem. Section 5 discusses convergence analysis for the approach. A few numerical examples and their results are provided in Section 6 to illustrate the method in action. The final section includes some concluding remarks.

2. Statement and properties of the problem

Considered a TPSPBVP of the form:

$$-\varepsilon\theta''(t)+\mu\alpha(t)\theta'(t)+\beta(t)\theta(t)=\gamma(t), \qquad 0<\varepsilon\ll 1,\ 0<\mu\ll 1, \tag{2.1}$$

$$\theta(0)=\xi,\ \theta(1)=\eta. \tag{2.2}$$

The functions $\alpha(t),\beta(t),\gamma(t)$ are sufficiently smooth with $\alpha(t)\geq\tilde{\alpha}>0$ and $\beta(t)\geq\tilde{\beta}>0$ in $[0,1]$, and ξ, η are finite constants. ε is the small positive perturbation parameter and μ is the small positive parameter. The roots of the characteristic equation:

$$-\varepsilon\lambda(t)^2+\mu\alpha(t)\lambda(t)+\beta(t)=0, \tag{2.3}$$

can be used to illustrate the solution of Eq.(2.1). It has two real solution functions given by:

$$\lambda_1(t)=\frac{\mu\alpha(t)}{2\varepsilon}-\sqrt{\left(\frac{\mu\alpha(t)}{2\varepsilon}\right)^2+\frac{\beta(t)}{\varepsilon}},$$

$$\lambda_2(t)=-\frac{\mu\alpha(t)}{2\varepsilon}+\sqrt{\left(\frac{\mu\alpha(t)}{2\varepsilon}\right)^2+\frac{\beta(t)}{\varepsilon}}. \tag{2.4}$$

Put $\tilde{\omega}_1 = -\max_{t\in[0,1]} \lambda_1 < \frac{-\mu}{\varepsilon} \le 0$ and $\tilde{\omega}_2 = \min_{t\in[0,1]} \lambda_2$. The solution decay in the boundary layer region is defined by $\tilde{\omega}_1$ and $\tilde{\omega}_2$.

For $\frac{\varepsilon}{\mu^2} \le 1$, $\left|\widetilde{\omega_1}\right| = O\left(\frac{\mu}{\varepsilon}\right)$ and $\left|\widetilde{\omega_2}\right| = O\left(\frac{1}{\mu}\right)$.

For $\frac{\mu^2}{\varepsilon} \le 1$, $\left|\widetilde{\omega_1}\right| = O\left(\frac{1}{\sqrt{\varepsilon}}\right)$ and $\left|\widetilde{\omega_2}\right| = O\left(\frac{1}{\sqrt{\varepsilon}}\right)$.

The layer at $t=0$ is defined by the term $e^{-\tilde{\omega}_1 t}$ and $t=1$ defined by the term $e^{-\tilde{\omega}_2(1-t)}$. As given in [13], we have:

$$\tilde{\omega}_1 = \begin{cases} \frac{\sqrt{\gamma\tilde{\alpha}}}{\sqrt{\varepsilon}}, & \frac{\mu}{\varepsilon^2} \le \frac{p}{\tilde{\alpha}} \\ \frac{\tilde{\alpha}\mu}{\varepsilon}, & \frac{\mu}{\varepsilon^2} \ge \frac{p}{\tilde{\alpha}} \end{cases} \text{ and } \tilde{\omega}_2 = \begin{cases} \frac{\sqrt{\gamma\tilde{\alpha}}}{2\sqrt{\varepsilon}}, & \frac{\mu}{\varepsilon^2} \le \frac{p}{\tilde{\alpha}} \\ \frac{p}{2\mu}, & \frac{\mu}{\varepsilon^2} \ge \frac{p}{\tilde{\alpha}} \end{cases}$$

where $\tilde{\alpha} = \min_{t\in[0,1]} \alpha(t)$ and $p = \min_{t\in[0,1]} \frac{\beta(t)}{\alpha(t)}$.

3. Description of spline method

Take a regular mesh with t_i as its nodes in $[0,1]$ such that $0 = t_0 < t_1 < \cdots < t_n = 1$, where $h = t_i - t_{i-1}$ for $i = 1,2,\ldots,N$. A function $S_i(t,\tau)$ of class $C^2[0,1]$ in the interval $[t_i, t_{i+1}]$ which interpolates $\theta(t)$ at discrete mesh t_i, depends on a parameter τ, transforms into a cubic spline $S_i(t)$, in $[0,1]$ as $\tau \to 0$ is named as an exponential spline.

Let $\theta(t)$ represent the accurate solution and θ_i represent an approximation to $\theta(t_i)$ based on the spline $S_i(t,\tau)$ passing through the points (t_i, θ_i) and (t_{i+1}, θ_{i+1}). In each i^{th} segment, the function $S_i(t,\tau)$ has the following form satisfying the first derivative continuity condition at the joint nodes (t_i, θ_i)

$$S_i(t) = L_i + M_i(t - t_i) + C_i e^{\tau(t-t_i)} + D_i e^{-\tau(t-t_i)}, \; i = 0,1,2,\ldots,N-1. \tag{3.1}$$

where L_i, M_i, C_i and D_i are constant coefficients, and τ is a free parameter that will be used to increase the efficiency of the scheme.

Define $S_i(t_i,\tau) = \theta_i$, $S_i(t_{i+1},\tau) = \theta_{i+1}$, $S_i''(t_i,\tau) = M_i, S_i''(t_{i+1},\tau) = M_{i+1}$ to calculate the coefficient values in Eq.(3.1) in terms of $\theta_i, \theta_{i+1}, M_i$, and M_{i+1}. Using the algebraic transformation, we get the following expression:

$$L_i = \theta_i - \frac{M_i}{\tau^2},\ M_i = \frac{\theta_{i+1}-\theta_i}{h} - \frac{(M_{i+1}-M_i)}{\tau\Theta},\ \ C_i = \frac{\left(M_{i+1} - e^{-\Theta}M_i\right)}{2\tau^2 \sinh(\Theta)},\ \ D_i = \frac{\left(e^{\Theta}M_i - M_{i+1}\right)}{2\tau^2 \sinh(\Theta)}$$

where $\Theta = \tau h$, for $i = 0, 1, ..., N-1$. With the help of continuity condition of the first derivative, i.e., $S'_{i-1}(t_i) = S'_i(t_i)$ at (t_i, θ_i), we obtain the subsequent relationships for $i = 1, 2, ..., N-1$.

$$\theta_{i-1} - 2\theta_i + \theta_{i+1} = h^2\left(\delta M_{i-1} + 2\lambda M_i + \delta M_{i+1}\right) \tag{3.2}$$

where $\delta = \dfrac{1}{\Theta^2} - \dfrac{1}{\Theta \sinh \Theta}$, $\lambda = \dfrac{\coth \Theta}{\Theta} - \dfrac{1}{\Theta^2}$, $M_j = \theta''(t_j),\quad j = i,\ i \pm 1$ and $\Theta = \tau h$.

The local truncation error $T_i(h)$ for the scheme proposed in Eq.(3.2) is:

$$T_i(h) = h^2\left(1 - 2(\delta + \lambda)\right)\theta''_i + h^4\left(\frac{1}{12} - \delta\right)\theta_i^{iv} + h^6\left(\frac{1}{360} - \frac{\delta}{12}\right)\theta_i^{vi} + \ldots + \infty \ \text{ for } \ i = 1, 2, ..., N-1.$$

As $\Theta \to 0, (\delta, \lambda) \to \left(\dfrac{1}{6}, \dfrac{2}{3}\right)$ then Eq.(3.1) reduces to ordinary cubic spline scheme. The scheme given in Eq.(3.2) is a tridiagonal scheme and consists of $(N-1)$ equations with $(N-1)$ unknowns $\theta_i, i = 1, 2, 3, \ldots, N-1.$

We have different orders for $T_i(h)$ for different values of α and λ in Eq.(3.2):

(i) fourth order for any arbitrary δ and β with $\delta + \lambda = \dfrac{1}{2}$,

(ii) sixth order for $\delta = \dfrac{1}{12}$, $\lambda = \dfrac{5}{12}$.

4. Description of method

At the point of grid t_i, the proposed scheme for the two-parameter singularly perturbed differential equation (2.1) may be discretized by:

$$\varepsilon M_i = \mu\alpha(t_i)\theta'(t_i) + \beta(t_i)\theta(t_i) - \gamma(t_i). \tag{4.1}$$

Using the above equations in Eq.(3.2) and applying the following first order derivative approximations of θ at the grid points $t_1, t_2, ..., t_{N-1}$,

$$\theta'_{i-1} \approx \frac{-\theta_{i+1} + 4\theta_i - 3\theta_{i-1}}{2h},\quad \theta'_{i+1} \approx \frac{3\theta_{i+1} - 4\theta_i + \theta_{i-1}}{2h},$$

$$\theta_i' \approx \left(\frac{1+2\psi h^2\beta_{i+1} + \psi h[3\alpha_{i+1} + \alpha_{i-1}]}{2h} \right)\theta_{i+1} - 2\psi[\alpha_{i+1} + \alpha_{i-1}]\theta_i -$$

$$+\left(\frac{1+2\psi h^2\beta_{i-1} - \psi h[\alpha_{i+1} + 3\alpha_{i-1}]}{2h} \right)\theta_{i-1} + \psi h[\gamma_{i+1} - \gamma_{i-1}],$$

we get the following tridiagonal system:

$$L_i\theta_{i-1} + C_i\theta_i + U_i\theta_{i+1} = W_i \quad \text{for} \quad i = 1, 2, \ldots, N-1 \tag{4.2}$$

where

$$L_i = -\varepsilon - \frac{3}{2}\delta\alpha_{i-1}h - \lambda\alpha_i h\left[1 + 2\psi h^2\beta_{i-1} - \psi h(\alpha_{i+1} + 3\alpha_{i-1})\right] + \frac{\delta}{2}\alpha_{i+1}h + \delta\beta_{i-1}h^2 ,$$

$$C_i = 2\varepsilon + 2\delta\alpha_{i-1}h - 4\lambda\alpha_i h^2\psi[\alpha_{i+1} + \alpha_{i-1}] - 2\delta\alpha_{i+1}h + 2\lambda\beta_i h^2 ,$$

$$U_i = -\varepsilon - \frac{\delta}{2}\alpha_{i-1}h + \lambda\alpha_i h\left[1 + 2\psi h^2\beta_{i+1} + \psi h(3\alpha_{i+1} + \alpha_{i-1})\right] + \frac{3}{2}\delta\alpha_{i+1}h + \delta\beta_{i+1}h^2 ,$$

$$W_i = h^2\left[(\delta + 2\psi\lambda\alpha_i h)\gamma_{i+1} + 2\lambda\gamma_i + (\delta - 2\psi\lambda\alpha_i h)\gamma_{i-1}\right],$$

$$\mu\alpha(t_i) = \alpha_i, \quad \beta(t_i) = \beta_i, \quad \gamma(t_i) = \gamma_i \quad \text{for } i = 0,\ 1,\ \ldots,\ N .$$

The tridiagonal system Eq.(4.2) is solved for $i = 1, 2, \ldots, N-1$ in order to obtain the approximations $\theta_1, \theta_2, \ldots, \theta_{N-1}$ of the solution $\theta(t)$ at $t_1, t_2, \ldots\ldots t_{N-1}$ .The local truncation error in Eq.(4.2) scheme is:

$$T_i(h) = \varepsilon\left[-1 + 2(\delta + \lambda)\right]h^2\theta'' + \left\{\left[\left(4\psi\varepsilon + \frac{1}{3}\right)\lambda - \frac{2\delta}{3}\right]\alpha_i\theta''' + (-1 + 12\delta)\frac{\varepsilon}{12}\theta^{iv}\right\}h^4 + O\left(h^6\right).$$

Hence, for various values of δ and λ, the truncation error has the following orders:

(i) For any arbitrary choice of δ, λ with $\delta + \lambda = \frac{1}{2}$ and for any value of ψ, $T_i(h)$ has fourth order.

(ii) For $\delta = \frac{1}{12}$, $\lambda = \frac{5}{12}$ and $\psi = -\frac{1}{20\varepsilon}$, $T_i(h)$ has sixth order.

5. Convergence analysis

In this section, we examine the convergence analysis of the proposed method. The following matrix of equations can be generated by including in boundary conditions:

$$(D + F)\theta + G + T(h) = 0 \tag{5.1}$$

where $D=\left[-\varepsilon,2\varepsilon,-\varepsilon\right]=\begin{bmatrix} 2\varepsilon & -\varepsilon & 0 & 0 & \ldots\ldots & 0 \\ -\varepsilon & 2\varepsilon & -\varepsilon & 0 & \ldots\ldots & 0 \\ 0 & -\varepsilon & 2\varepsilon & -\varepsilon & \ldots\ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & .. & .. & \ldots & \ldots \\ 0 & .. & .. & 0 & -\varepsilon & 2\varepsilon \end{bmatrix}$

and

$$F=\left[\tilde{z}_i,\tilde{v}_i,\tilde{w}_i\right]=\begin{bmatrix} \tilde{v}_1 & \tilde{w}_1 & 0 & 0 & \ldots\ldots & 0 \\ \tilde{z}_2 & \tilde{v}_2 & \tilde{w}_2 & 0 & \ldots\ldots & 0 \\ 0 & \tilde{z}_3 & \tilde{v}_3 & \tilde{w}_3 & \ldots\ldots & 0 \\ .. & .. & .. & .. & .. & .. \\ .. & .. & .. & .. & .. & .. \\ 0 & .. & .. & 0 & \tilde{z}_{N-1} & \tilde{v}_{N-1} \end{bmatrix}$$

where

$$\tilde{z}_i=-\frac{3}{2}\delta\alpha_{i-1}h-\lambda\alpha_i h\left[1+2\psi h^2\beta_{i-1}-\psi h\left(\alpha_{i+1}+3\alpha_{i-1}\right)\right]+\frac{\delta}{2}\alpha_{i+1}h+\delta\beta_{i-1}h^2 ,$$

$$\tilde{v}_i=2\delta\alpha_{i-1}h-4\lambda\alpha_i h^2\psi\left[\alpha_{i+1}+\alpha_{i-1}\right]-2\delta\alpha_{i+1}h+2\lambda\beta_i h^2 ,$$

$$\tilde{w}_i=-\frac{\delta}{2}\alpha_{i-1}h+\lambda\alpha_i h\left[1+2\psi h^2\beta_{i+1}+\psi h\left(3\alpha_{i+1}+\alpha_{i-1}\right)\right]+\frac{3}{2}\delta\alpha_{i+1}h+\delta\beta_{i+1}h^2$$

and

$$G=\left[q_1-\tilde{z}_1\xi,q_2,q_3,\ldots,q_{N-1}-\tilde{w}_{N-1}\eta\right]$$

where

$$q_i=h^2\left[\left(\delta+2\psi\lambda\alpha_i h\right)\gamma_{i+1}+2\lambda\gamma_i+\left(\delta-2\psi\lambda\alpha_i h\right)\gamma_{i-1}\right],\ \text{for}\ i=1,2,3,\ldots,N-1$$

and

$$T(h)=O\left(h^6\right)\ \text{for}\ \delta=\frac{1}{12},\ \lambda=\frac{5}{12}\ \text{and}\ \psi=-\frac{1}{20\varepsilon}\ \text{and}\ \theta=\left[\theta_1,\theta_2,\ldots,\theta_{N-1}\right]^T ,$$

$$T_i(h)=\left[T_1,T_2,\ldots,T_{N-1}\right]^T ,O=\left[0,0,\ldots,0\right]^T$$ associated vectors of Eq.(5.1).

Let the vector $\varphi=\left[\varphi_1,\varphi_2,\ldots,\varphi_{N-1}\right]^T\cong\theta$ that satisfies the equation:

$$\left(D+F\right)\varphi+G=0 . \tag{5.2}$$

Let $e_i=\varphi_i-\theta_i,\ i=1(1)N-1$ be the error corresponding to the discretization so that:

$$E=\left[e_1,e_2,...,e_{N-1}\right]^T=\varphi-\theta.$$

Subtracting Eq.(5.1) from Eq.(5.2), we obtain the error equation:

$$(D+F)E=T(h). \tag{5.3}$$

Let $|\alpha(t)|\le\xi_1$ and $|\beta(t)|\le\xi_2$ where ξ_1,ξ_2. are positive constants. If $F_{i,j}$ be the $(i,j)^{th}$ element of F, then

$$\left|F_{i,i+1}\right|=\left|\tilde{w}_i\right|\le\left(h(\delta+\lambda)\xi_1+h^2\delta\xi_2+4\lambda\psi h^2\xi_1^2+2h^3\lambda\psi\xi_1\xi_2\right),\ \text{for}\ i=1,2,3,\dots,N-2,$$

$$\left|F_{i,i-1}\right|=\left|\tilde{z}_i\right|\le h(\delta+\lambda)\xi_1+h^2\delta\xi_2+4\lambda\psi h^2\xi_1^2+2h^3\lambda\psi\xi_1\xi_2,\ \text{for}\ i=2,3,4,\dots,N-1.$$

Thus, for a sufficiently small h, we have:

$$\left|F_{i,i+1}\right|<\varepsilon,\quad i=1,2,\dots,N-2. \tag{5.4}$$

$$\left|F_{i,i-1}\right|<\varepsilon,\quad i=2,3,\dots,N-1. \tag{5.5}$$

Let $\tilde{S}_i$ be the sum of the i^{th} row elements of the matrix $(D+F)$, then we have:

$$\tilde{S}_i=\varepsilon-\frac{\delta h}{2}\left(\alpha_{i+1}-3\alpha_{i-1}\right)+h\lambda\alpha_i+h^2\left(\delta\beta_{i-1}+2\lambda\beta_i\right)+$$
$$+h^2\lambda\psi p_i\left(3\alpha_{i-1}+\alpha_{i+1}\right)-2h^3\lambda\psi\alpha_i\beta_{i-1}\quad\text{for}\quad i=1,$$

$$\tilde{S}_i=h^2\left(\beta_{i-1}+2\lambda\beta_i+\delta\beta_{i+1}\right)+2h^3\lambda\alpha_i\psi\left(\beta_{i+1}-\beta_{i-1}\right)\ \text{for}\ i=2,3,\dots,N-2,$$

$$\tilde{S}_i=\varepsilon+\frac{\delta h}{2}\left(\alpha_{i-1}-3\alpha_{i+1}\right)-h\lambda\alpha_i+h^2\left(\delta\beta_{i-1}+2\lambda\beta_i\right)-$$
$$+h^2\lambda\psi\alpha_i\left(3\alpha_{i+1}+\alpha_{i-1}\right)-2h^3\lambda\psi\alpha_i\beta_{i-1}\quad\text{for}\quad i=N-1.$$

Let $\xi_{1^*}=\min\limits_{1\le i\le N}\left|p(t_i)\right|$ and $\xi_1^*=\max\limits_{1\le i\le N}\left|p(t_i)\right|$, $\xi_{2^*}=\min\limits_{1\le i\le N}\left|q(t_i)\right|$ and $\xi_2^*=\max\limits_{1\le i\le N}\left|q(t_i)\right|$. Since $0<\varepsilon\ll 1$ and $\varepsilon\propto O(h)$, it is verified that $(D+F)$ is monotone [16, 17] with sufficiently small h.

Therefore $(D+F)^{-1}$ exists and $(D+F)^{-1}\ge 0.$ So, from Eq.(5.5), we get:

$$\|E\|\le\left\|(D+F)^{-1}\right\|\|T\|. \tag{5.6}$$

Let $(D+F)^{-1}_{i,k}$ be the $(i,k)^{th}$ element of $(D+F)^{-1}$ and we define:

$$\left\|(D+F)^{-1}\right\| = \max_{1\le i\le N-1} \sum_{k=1}^{N-1} (D+F)^{-1}_{i,k} \text{ and } \left\|T(h)\right\| = \max_{1\le i\le N-1} \left|T(h)\right|. \tag{5.7}$$

Since $(D+F)^{-1}_{i,k} \ge 0$ and $\sum_{k=1}^{N-1} (D+F)^{-1}_{i,k} \cdot \tilde{S}_k = 1$ for $i = 1, 2, 3, \ldots\ldots, N-1$.

Hence:

$$(D+F)^{-1}_{i,1} \le \frac{1}{\tilde{S}_1} < \frac{1}{h^2\left[(\delta+2\beta)\xi_{2^*} - 4\beta\psi\xi_1^2\right]}, \tag{5.8}$$

$$(D+F)^{-1}_{i,N-1} \le \frac{1}{\tilde{S}_{N-1}} < \frac{1}{h^2\left[(\delta+2\beta)\xi_{2^*} - 4\beta\psi\xi_1^2\right]}. \tag{5.9}$$

Furthermore,

$$\sum_{k=2}^{N-2} (D+F)^{-1}_{i,k} \le \frac{1}{\min\limits_{2\le k\le N-2} \tilde{S}_k} < \frac{1}{h^2\left[2(\delta+\beta)\xi_{2^*}\right]}, \quad i = 2, 3, \ldots, N-2. \tag{5.10}$$

By the help of Eqs. (5.7) - (5.10), from (5.6), we obtain:

$$\left\|E \le O\left(h^4\right)\right\|. \tag{5.11}$$

Thus, the method given by Eq.(5.10) is convergent in fourth order for

$$\delta = \frac{1}{12}, \ \beta = \frac{5}{12} \text{ and } \psi = -\frac{1}{20\varepsilon}.$$

6. Numerical examples

To illustrate the computational feasibility of our proposed method based on an exponential spline function, three TPSPBVPs are considered. These problems were selected because they have received extensive attention in the literature and have well-defined solutions that may be used as comparisons. Maximum absolute errors [MAE] for the problems under consideration have been tabulated and compared to existing methods to demonstrate the superiority of the proposed method.

Example 1. $\varepsilon\theta'' + \mu\theta' - \theta = -t, \ 0 < t < 1$ with $\theta(0) = 1, \ \theta(1) = 0$.

The exact solution is given by:

$$\theta(t)=\frac{(1+\mu)+(1-\mu)\exp(\lambda_2)}{\exp(\lambda_2)-\exp(\lambda_1)}\exp(\lambda_1 t)+\frac{(1+\mu)+(1-\mu)\exp(\lambda_1)}{\exp(\lambda_1)-\exp(\lambda_2)}\exp(\lambda_2 t)+t+\mu$$

where

$$\lambda_1=\frac{\mu-\sqrt{\mu^2+4\varepsilon}}{2\varepsilon},\quad \lambda_2=\frac{\mu+\sqrt{\mu^2+4\varepsilon}}{2\varepsilon}.$$

MAEs for different values of ε, μ and h are shown in Tab.1 and Tab.2.

Example 2. $-\varepsilon\theta''+\mu\theta'+\theta=\cos(\pi t),\ \ 0<t<1$ with $\theta(0)=1,\ \theta(1)=0$.

The exact solution is $\theta(t)=\rho_1\cos(\pi t)+\rho_2\sin(\pi t)+A_1\exp(\lambda_1 t)+A_2\exp\left[-\lambda_1(1-t)\right]$, where

$$\rho_1=\frac{\varepsilon\pi^2+1}{\varepsilon^2\pi^2+\left(\varepsilon\pi^2+1\right)^2},\qquad \rho_2=\frac{\varepsilon\pi}{\varepsilon^2\pi^2+\left(\varepsilon\pi^2+1\right)^2},\qquad A_1=-\rho_1\frac{1+exp(-\lambda_2)}{1-exp(\lambda_1-\lambda_2)},$$

$$A_2=\rho_2\frac{1+exp(\lambda_2)}{1-exp(\lambda_1-\lambda_2)},\qquad \lambda_1=\frac{\mu-\sqrt{\mu^2+4\varepsilon}}{2\varepsilon},\quad \lambda_2=\frac{\mu+\sqrt{\mu^2+4\varepsilon}}{2\varepsilon}.$$

MAEs for different values of ε,μ and h are shown in Tab.3.

Example 3. $-\varepsilon\theta''-2\mu\theta'+4\theta=1,\ \ 0<t<1$ with $\theta(0)=0,\ \theta(1)=1$.

The maximum absolute errors are obtained using the principle of double mesh, $E_i^N=\max_{0\le i\le N}\left|\nu_i^N-\nu_{2i}^{2N}\right|$. MAEs for different values of ε and h are shown in Tab.4 and Tab.5.

7. Conclusion

In this article, a fourth-order numerical approach for solving a class of TPSPBVP is developed using the spline method. To solve the difference method's tridiagonal scheme, discrete invariant embedding was used to solve the resulting tridiagonal difference scheme from the suggested scheme. Three standard test examples from the literature were considered, and the maximum absolute errors of their solutions were presented in Tabs 1-5. Graphical representations are used to show how small parameters affect the behaviour of the solution (Figs 1-6). We observed that as ε declined for a fixed μ, the width of the left and right boundary layer decreases. For a fixed ε, when μ decreases the width of the left layer increases and right layer decreases. The numerical solutions also correspond extremely well to the exact solutions. Finally, it was noticed that the proposed fourth-order scheme produced better outcomes than the previous methods, proving the superiority of the suggested method.

Table 1. MAE and rate of convergence in Example 1 for $\varepsilon = 10^{-3}$.

$\mu \downarrow N \rightarrow$	64	128	256	512	1024
			Results in [18]		
10^{-2}	3.6590e-03	1.1005e-03	2.7573e-04	6.8812e-05	1.7196e-05
	1.7332	1.9968	2.0025	2.0025	--
10^{-3}	3.0262e-03	7.4023e-04	1.8406e-04	4.5953e-05	1.1484e-05
	2.0314	2.0077	2.0019	2.0005	--
10^{-4}	2.9008e-03	7.0989e-04	1.7654e-04	4.4076e-05	1.1015e-05
	2.0307	2.0076	2.0019	2.0005	--
			Our method		
10^{-2}	4.7349e-005	3.0188e-006	1.8909e-007	1.1840e-008	7.3990e-010
	3.9713	3.9968	3.9973	4.0002	--
10^{-3}	4.5944e-005	2.8914e-006	1.8104e-007	1.1320e-008	7.0776e-010
	3.9900	3.9974	3.9994	3.9995	--
10^{-4}	4.5331e-005	2.8528e-006	1.7862e-007	1.1169e-008	6.9796e-010
	3.9900	3.9974	3.9993	4.0002	--

Table 2. MAE and rate of convergence in Example - 1 for $\mu = 10^{-4}$.

$\varepsilon \downarrow N \rightarrow$	64	128	256	512	1024
			Results in [18]		
10^{-1}	1.5752e-05	3.9408e-06	9.8514e-07	2.4628e-07	6.1570e-05
	1.9990	2.0001	2.0000	2.0000	--
10^{-2}	2.8064e-04	7.0125e-05	1.7522e-05	4.3807e-06	1.0952e-06
	2.0007	2.0008	1.9999	2.0000	--
10^{-3}	2.9008e-03	7.0989e-04	1.7654e-04	4.4076e-05	1.1015e-05
	2.0307	2.0076	2.0019	2.0005	--
			Our method		
10^{-1}	2.5619e-009	1.6033e-010	1.0225e-011	1.5884e-012	4.5530e-012
	3.9981	3.9709	3.9973	4.0002	--
10^{-2}	4.5465e-007	2.8485e-008	1.7806e-009	1.1131e-010	7.0776e-010
	3.9900	3.9998	3.9997	3.9994	--
10^{-3}	4.5331e-005	2.8528e-006	1.7862e-007	1.1169e-008	6.9796e-010
	3.9900	3.9974	3.9993	4.0002	--

Table 3. MAE in Example 2.

$\varepsilon = 10^{-4}$, $N = 128$				
μ	Results in [11]	Results in [14]	Results in [19]	Our Result
10^{-3}	0.94446e-002	0.47598e-002	0.51964e-002	0.27811e-003
10^{-4}	0.90436e-002	0.42856e-002	0.41710e-002	0.27240e-003
10^{-5}	0.90036e-002	0.42295e-002	0.40754e-002	0.27148e-003
10^{-6}	0.89996e-002	0.42238e-002	0.40659e-002	0.27139e-003
10^{-7}	0.89992e-002	0.42232e-002	0.40650e-002	0.27138e-003

Table 4. MAE and rate of convergence in Example 3 for $\varepsilon = 10^{-3}$

$\mu \downarrow N \rightarrow$	64	128	256	512	1024
			Results in [18]		
10^{-2}	3.6016e-03	1.6211e-03	7.0696e-04	2.8171e-04	1.0312e-04
	1.1516	1.1972	1.3274	1.4498	--
10^{-3}	3.7743e-03	1.5538e-03	5.9114e-04	2.0815e-04	6.2089e-05
	1.2804	1.3942	1.5058	1.7452	--
10^{-4}	3.7827e-03	1.5541e-03	5.8893e-04	2.0559e-04	5.9990e-05
	1.2833	1.3999	1.5183	1.7769	--
			Our method		
10^{-2}	5.1235e-004	3.2945e-005	2.1013e-006	1.3163e-007	8.2422e-009
	3.9590	3.9707	3.9967	3.9973	--
10^{-3}	5.0365e-004	3.2257e-005	2.0307e-006	1.2716e-007	7.9513e-009
	3.9647	3.9896	3.9973	3.9993	--
10^{-4}	4.9741e-004	3.1856e-005	2.0054e-006	1.2557e-007	7.8521e-009
	3.9648	3.9896	3.9973	3.9993	--

Table 5. MAE and rate of convergence in Example 3 for $\mu = 10^{-4}$.

$\varepsilon\downarrow N\rightarrow$	64	128	256	512	1024
			Results in [18]		
10^{-1}	8.2643e-05	2.0643e-05	5.1596e-06	1.2899e-06	3.2247e-07
	2.0012	2.0003	2.0000	2.0000	--
10^{-2}	8.4861e-04	2.1035e-04	5.2577e-05	1.3137e-05	3.2843e-06
	2.0123	2.0002	2.0007	1.9999	--
10^{-3}	3.7827e-03	1.5541e-03	5.8893e-04	2.0559e-04	5.9990e-05
	1.2833	1.3999	1.5183	1.7769	--
			Our method		
10^{-1}	5.0396e-008	3.1508e-009	1.9659e-010	1.1526e-011	1.0001e-012
	3.9995	4.0025	4.0922	3.5267	--
10^{-2}	5.1107e-006	3.2037e-007	2.0078e-008	1.2550e-009	7.8151e-011
	3.9957	3.9961	3.9999	4.0053	--
10^{-3}	4.9741e-004	3.1856e-005	2.0054e-006	1.2557e-007	7.8521e-009
	3.9648	3.9896	3.9973	3.9993	--

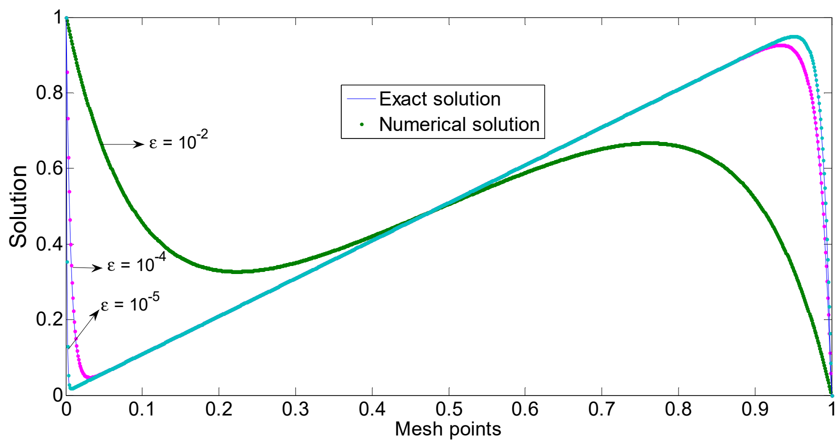

Fig.1. Solution profile of example 1 with $\mu = 10^{-2}$.

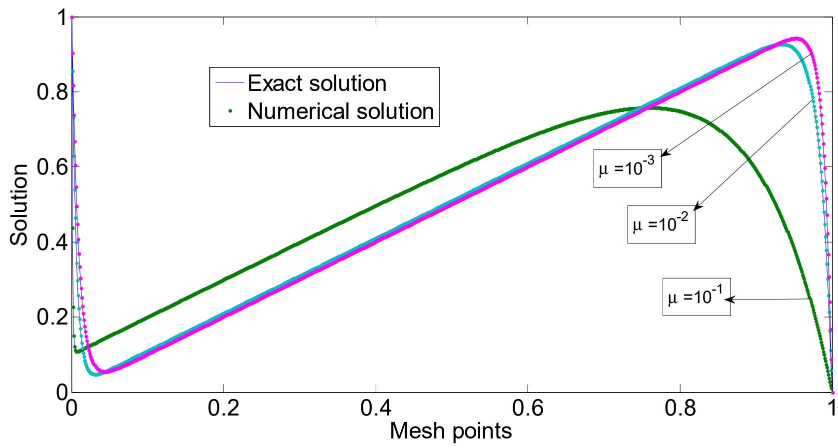

Fig.2. Solution profile of example 1 with $\varepsilon = 10^{-4}$.

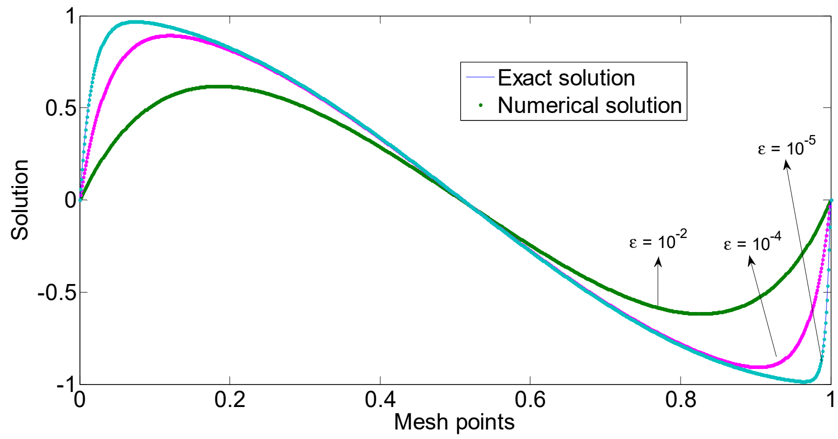

Fig.3. Solution profile of example 2 for $\mu = 10^{-2}$.

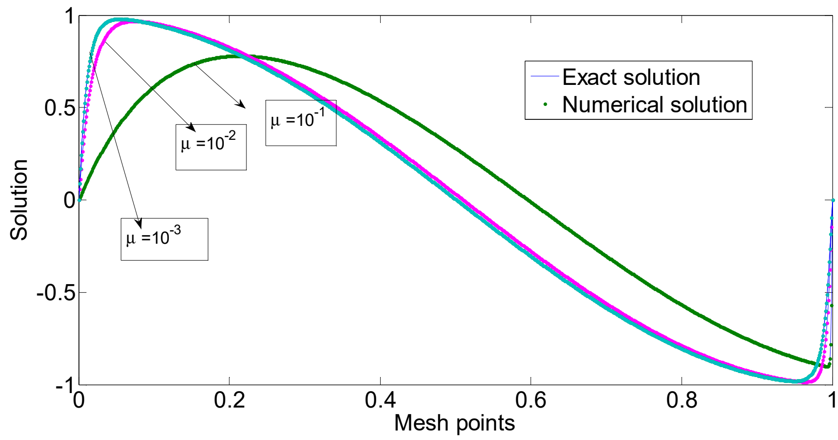

Fig.4. Solution profile of example -2 for $\varepsilon = 10^{-4}$.

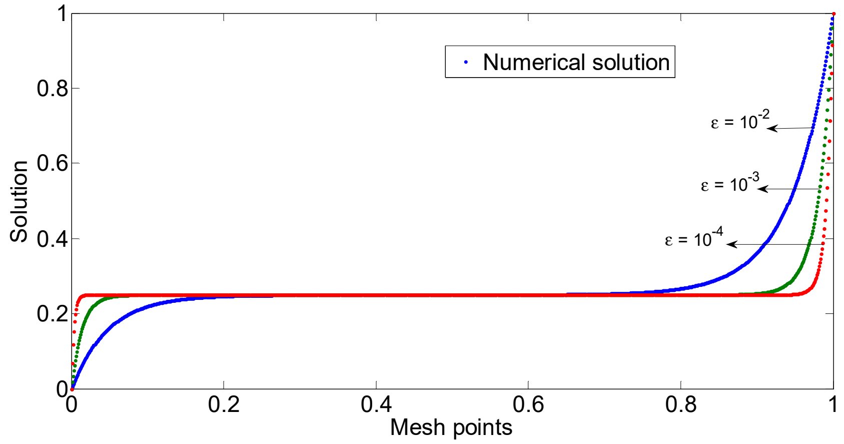

Fig.5. Solution profile of example 3 for $\mu = 10^{-2}$.

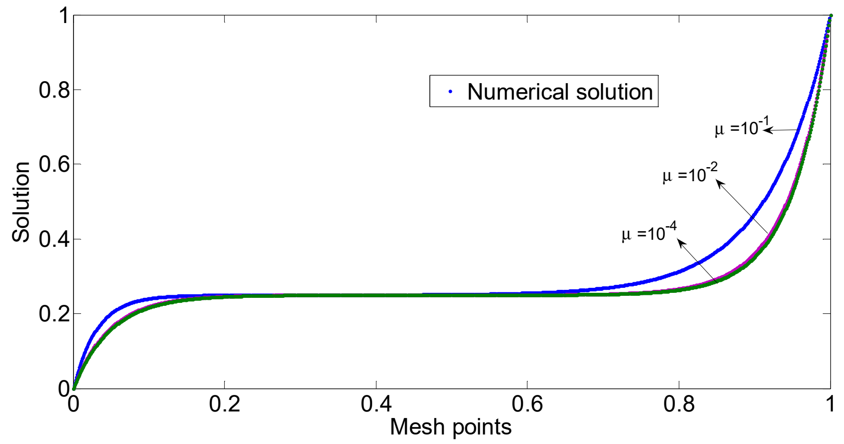

Fig.6. Solution profile of example 3 for $\varepsilon = 10^{-3}$, $\mu = 10^{-2}$.

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Nomenclature

E	– error matrix
E^N	– absolute error
h	– mesh size
N	– number of sub-intervals
$T_i(h)$	– truncation error
t	– independent variable
t_i	– mesh points
$\tilde{\alpha}$	– positive constant
$\alpha(t)$, $\beta(t)$, $\gamma(t)$	– smooth functions
ε	– perturbation parameter

θ – solution
μ – small positive parameter
ξ, η – finite constants
τ – free parameter

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