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Alternating projection-based iterative learning control for discrete-time systems with non-uniform trial lengths

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Summary

This paper develops a novel framework for iterative learning control (ILC) design of discrete-time systems with non-uniform trial lengths by using alternating projections. In contrast to existing works for the non-uniform trial length problem, this paper uses the Hilbert space settings and hence the discrete-time system dynamics with non-uniform trial lengths can be represented by multiple affine subspaces (or linear varieties). Motivated by the successive projection design between two closed convex sets, the considered ILC problem can be transformed into alternating projections between multiple sets, then well-defined convergence properties can be derived. Moreover, an optimal ILC design is produced for systems with non-uniform trial lengths under the proposed framework, which is also extended to the case of input constraints. A numerical case study is given to illustrate the applicability of the new design.

KEYWORDS:
iterative learning control, discrete-time system, alternating projection, non-uniform trial length, input constraint

1 | INTRODUCTION

Iterative learning control (ILC) is applicable to systems that repeatedly complete the same finite duration task. An example is a pick-and-place robot performing the following steps: i) collect the payload from a specified location, ii) transfer it over a finite duration, iii) place the payload on a moving conveyor under synchronization, iv) return to the starting location, and v) repeat these steps as many times as possible. Let the finite duration be termed the trial length and use the term trial to denote each execution. For discrete dynamics, the notation for the system output is $y_k(t)$, $t \in [1,\ N]$, where both discrete time instant t and trial number k are non-negative integers, and $N < \infty$ denotes the trial length.

Once a trial is complete, all information generated, i.e., for $t \in [0,\ N]$, is available for use in updating the control signal for the next trial. Let $u_k(t),\ t \in [0,\ N-1],\ k \geq 0$, denote the control input on trial k. Then in the ILC setting, suppose that $y_d(t)$, $t \in [1,\ N]$, is a supplied reference trajectory, e.g., the desired path to be followed by the pick-and-place robot on each trial. In which case the sequence of errors $\{e_k\}_{k\geq 0}$ can be formed, where on trial k, $e_k(t) = y_d(t) - y_k(t)$.

The ILC design problem can now be formulated as the construction of a control input sequence $\{u_k\}_{k\geq 0}$ that enforces convergence along the trial k under an appropriate norm to either zero in the ideal case, or to within some acceptable bounds. In ILC, the input is regulated and one form of control law is to construct the control input for the next trial as the sum of that used on the previous trial and a correction based on previous trial information (in some cases a current trial feedback term is also added). The critical feature in ILC is that all information from the previous trial is available to the control law. Hence, for example, an ILC phase-lead law has the structure $u_{k+1}(t) = u_k(t) + \omega e_k(t+\beta),\ \beta \geq 0$, where the non-negative integer β denotes the phase-lead.

The phase-lead term in this last ILC law is implementable because it acts on the previous trial error. If $\beta = 0$, it can be shown that an equivalent feedback control law exists and ILC has no added benefit. Since the mid-1980s, in particular, ILC has remained an active research area, e.g., the first work on robotics [1] and the survey papers. [2,3] A strong feature of the research is the number of design algorithms that have proceeded to a least experimental validation. Engineering applications include multi-agent systems, [4,5] printing systems [6] and center-articulated vehicles. [7] Also in the process industries with batch processes, see, e.g., batch process. [8,9]

In the great majority of the ILC literature, the systems are required to track a desired reference trajectory of a fixed length and specified a priori. An application area, where variable or non-uniform trial lengths arise, is in the use of ILC to regulate the level of stimulation applied to patients undergoing robotic-assisted stroke rehabilitation. People who suffer a stroke loose functionality down one side of their body and the recommended method of attempting to recover lost functionality is repeated attempts at a task, e.g., reaching out to an object. However, patients cannot move the affected limb and the quality of rehabilitation is poor.

Muscles can be made to move by application of electrical stimulation to the muscles involved, but there is a need to tightly regulate the applied stimulation to achieve maximum effect. In previous work for the upper limb, it was established in the work of Freeman et al, [10] with supporting clinical trials, [11] that ILC can be deployed to regulate the stimulation, where if the patient is improving with each attempt, the voluntary effort should increase and the applied stimulation decrease. Exactly this effect was detected in the clinical trials.

In the early stroke rehabilitation work, the reference trajectory is chosen based on a healthcare professionals interpretation of the patients current ability, and must not be too hard (loss of motivation often results) or too easy (no benefit from the session).

This early work led on to other work on the use of ILC in healthcare, where the trial length is fixed. One area is the use of ILC-based functional electrical stimulation to help stroke patients to recover from physiological foot motion. However, for safety reasons or unpredictable voluntary effort by the patient, the stimulation signal must be applied until the initial contact was detected between foot and the ground, which gives rise to an ILC problem of design for non-identical trial lengths.[12] Another area where non-identical, termed non-uniform from this point onwards, trial lengths occur, lies in the filling phase of the injection molding.[13] The filling phase should be switched to the next phase instantly once the pressure in the molding chamber increases to a certain value, and thus non-uniform trial lengths arise. In response, there has been research on ILC design for non-uniform trial lengths.

Early research approaches to this last problem include an iterative average operator for improving learning performance by utilizing the historical error and input information.[14,15,16] However, the historical information may be redundant and affects the utilization of up-to-date information. In the work of Shen et al,[17] a lifted framework, also called intermittent ILC,[18] was developed for systems with randomly varying trial lengths using the P-type ILC law, in which stronger convergence results in random sense were obtained. In contrast to the random model for varying trial lengths, a deterministic convergence property was studied in the work of Meng and Zhang[19] for tracking design in the presence of non-uniform trial lengths. Also, a necessary and sufficient condition for monotonic convergence was developed for simple structure ILC design for systems with non-uniform trial lengths in the work of Seel et al.[20] It was reported in the work of Jin[21] that a design where only information of the most recent trial is learned and a modified composite energy function is employed to analyze the convergence property. Moreover, a robust ILC scheme combined with adaptive design techniques was proposed to handle non-uniform trial length systems with nonparametric uncertainties.[22] It was also reported in the work of Shen et al[23] that an adaptive ILC scheme is built for the case under state alignment condition with varying trial lengths by resorting to a barrier composite energy function approach. However, aforementioned approaches mainly focus on the improvement of learning efficiency, which cannot definitely increase the convergence speed. A critical question in ILC performance is the speed of the error convergence and how to increase or accelerate it if required. In this sense, the norm optimal ILC was modified for handling the non-uniform trial length problem existing in the application of ventricular assist devices.[24]. However, there is no theoretical proof about the feasibility of optimal ILC applied to the non-uniform trial lengths in this work. Also, an intermittent optimal learning control scheme was developed in the work of Liu et al,[25] which is intended to fast convergence speed by minimizing a designed performance index.

Moreover, how to conduct the convergence analysis of these ILC approaches for non-uniform trial length problem is also one of main issues. Some effective methods have been introduced to deal with this difficulty, including contraction mapping method[14,16,26] and Lyapunov-based composite energy function method.[21,22,23] Different from existing methods, this paper proposes a new design and analysis framework for discrete-time systems with non-uniform trial lengths. This framework is based on the method of alternating projections in Hilbert space, and hence naturally handles this issue from an optimization point of view. The introduction of Hilbert space setting can simplify the design and analysis of complex ILC control problem by using the language of operator theory.[27] The alternating projection method to ILC, also termed successive projection,[28,29] has been developed for constant trial length systems, which has been extended to the non-uniform trial length case by utilizing the mathematical expectation for employing only two projecting sets.[30] Different from this previous work, this paper aims at extending the alternating projection ILC framework to the non-uniform trial length case by introducing more than two sets, namely, finite number of sets, so as to obtain stronger convergence results in a deterministic way. This is the main motivation of this paper. It should be pointed that that alternating projections between multiple sets would not decrease the convergence speed compared to that between two sets if the projection orders are reasonable designed.

To conclude, this paper develops an optimal ILC design using the modified alternating projection framework for discrete-time multiple-input multiple-output (MIMO) systems with non-uniform trial lengths. Multiple affine subspaces (or linear varieties) are employed to represent the discrete-time system dynamics with non-uniform trial lengths, then the result of alternating projections between multiple sets can be utilized for the optimal ILC design and convergence analysis. In this case, the causal implementation is allowed by using the norm optimal setting with modifications. Furthermore, the ILC design is extended to the case with non-uniform trial lengths where input constraints arise or must be imposed for applications specific reasons. In all cases, the error convergence properties are analyzed. Finally, a numerical case study based on a model obtained from a coarse-fine stage is given to demonstrate the applicability of the new design.

The major novel contributions of this paper are as follows:

- An ILC design framework is developed for discrete-time systems with non-uniform trial lengths by using the method of alternating projections.
- A practical causal feedback plus feedforward design for discrete-time systems with non-uniform trial lengths is developed, whose convergence is theoretically proved.
- The proposed alternating projection framework is extended to the design for non-uniform trial length case with input constraints.

The structure of this paper is organized as below. The problem formulation is first addressed in Section 2. Section 3 develops an ILC design for non-uniform trial lengths using alternating projections, and a causal feedback plus feedforward structure is derived for practical implementation. Section 4 gives the new results for input constraints. A numerical case study is carried out in Section 5, and the conclusions are given in Section 6.

Throughout this paper, $\mathbb{N}$ denotes the set of natural number; $\mathbb{R}^n$ and $\mathbb{R}^{n\times m}$ denote the sets of n-dimensional real vectors and $n \times m$ real matrices, respectively; $l_2^{\mathrm{m}}[a, b]$ denotes the space of $\mathbb{R}^{\mathrm{m}}$ valued Lebesgue square-summable sequences defined on an interval $[a, b]$; The superscripts T and $\perp$ respectively denote the transpose and the orthogonal complement operations; $x \perp y$

represents that x and y are orthogonal; $\mathbf{0}$ denotes zero vector with compatible dimensions; $P_M(x)$ denotes the projection of x onto the set M in a Hilbert space; $\bigcap$ denotes the intersection of sets; $\langle\cdot\rangle$ denotes the inner product, and $\mathbb{X}\times\mathbb{Y}$ denotes the Cartesian product of two spaces $\mathbb{X}$ and $\mathbb{Y}$. Other notations will be introduced when required.

2 | PROBLEM FORMULATION

Consider a linear time-invariant discrete-time MIMO system with non-uniform trial lengths described in the ILC setting by the state-space model

$$\begin{cases} x_k(t+1) = Ax_k(t) + Bu_k(t), \\ y_k(t) = Cx_k(t), \end{cases} \tag{1}$$

where $k \in \mathbb{N}$ and $t \in [0,\ N_k]$ respectively denote the trial number and time index. N_k is a random variable that represents the actual trial length of trial k. $x_k(t) \in \mathbb{R}^{\mathrm{n}}$, $u_k(t) \in \mathbb{R}^{\ell}$ and $y_k(t) \in \mathbb{R}^{\mathrm{m}}$ denote the state, input and output of system (1), respectively. It is assumed that CB is full-rank so that the relative degree is equal to one. Without loss of generality, it is also assumed that $x_k(0) = x_0$ for all trials, i.e., same state initial vector on each trial.

One method for ILC analysis and design for the systems considered is to use the lifted model representation, where the values of a variable are represented by assembling them in order as the entries in a vector, and this vector has a finite dimension due to the finite trial length. In this approach, the error updating from trial-to-trial is governed by a standard difference equation and analyzed by standard discrete linear systems theory. See, e.g., the survey papers,[2,3] for the background on this approach to ILC design.

The non-uniform trial length case does not follow as a direct generalization of lifted model. Therefore, the actual trial length N_k is set to vary in $\{N_m, N_m+1, \ldots, N\}$, where N_m and N respectively denote the minimum and maximum trial lengths that occur in a particular application, for which there are $J = N - N_m + 1$ possible trial lengths. In this case, the lifted model with same trial length N can be employed, i.e.

$$y_k = Gu_k + d_k, \tag{2}$$

where G and d_k represent the system model and the effect of the initial conditions respectively, i.e.

$$G = \begin{bmatrix} CB & 0 & 0 & \cdots & 0 \\ CAB & CB & 0 & \cdots & 0 \\ CA^2B & CAB & CB & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ CA^{N-1}B & CA^{N-2}B & CA^{N-3}B & \cdots & CB \end{bmatrix}, \tag{3}$$

$$d_k = \left[(CA)^T \ \left(CA^2\right)^T \ \cdots \ \left(CA^N\right)^T \right]^T x_k(0), \tag{4}$$

and

$$u_k = \left[u_k^T(0), u_k^T(1), \ldots, u_k^T(N-1)\right]^T, \tag{5}$$

$$y_k = \left[y_k^T(1), y_k^T(2), \ldots, y_k^T(N)\right]^T. \tag{6}$$

This paper uses a Hilbert space setting. Let $l_2^{\ell}[0,\ N-1]$ and $l_2^{\mathrm{m}}[1,\ N]$ denote the input and output spaces respectively, with inner products and associated induced norms

$$\langle u, v\rangle_R = \sum_{i=0}^{N-1} u^T(i)\, Rv(i), \quad \|u\|_R = \sqrt{\langle u, u\rangle_R}, \tag{7}$$

$$\langle y, e\rangle_Q = \sum_{i=1}^{N} y^T(i)\, Qe(i), \quad \|y\|_Q = \sqrt{\langle y, y\rangle_Q}, \tag{8}$$

where $u, v \in l_2^{\ell}[0,\ N-1]$ and $y, e \in l_2^{\mathrm{m}}[1,\ N]$, and $R \in \mathbb{R}^{\ell\times\ell}$ and $Q \in \mathbb{R}^{\mathrm{m}\times\mathrm{m}}$ are symmetric positive definite weighting matrices. Define $y_d(t)$ as the desired output or reference trajectory for $t \in [1,\ N]$ in the lifted model setting, i.e.

$$y_d = \left[y_d^T(1), y_d^T(2), \ldots, y_d^T(N)\right]^T. \tag{9}$$

One problem for ILC design in the non-uniform trial length case is that the actual output values in y_k on trial k are not known for $t \in [N_k+1,\ N]$. The reference trajectory is, however, known and hence it is possible to set

$$y_k(t) = y_d(t), \ \ t \in [N_k+1,\ N]. \tag{10}$$

In this way, learning efficiency of the lifted model for systems with non-uniform trial lengths along the trial can be maintained when using the tracking error to update the input signal for the next trial.

To describe the tracking error of systems with non-uniform trial lengths, a trial-varying matrix is introduced as

$$F_k = \begin{bmatrix} \boldsymbol{I}_{N_k} \otimes \boldsymbol{I}_{\mathrm{m}} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \otimes \boldsymbol{I}_{\mathrm{m}} \end{bmatrix}, \tag{11}$$

where I_l denotes the identity matrix with dimensions $l \times l$, and $\otimes$ denotes the Kronecker product. Then, the tracking error can be written as

$$e_k = F_k\left(y_d - y_k\right), \tag{12}$$

even though the output at $t \in [N_k+1,\ N]$ is unknown. In this sense, the error vectors for different trials belong to different subspaces, which depends on the trial number.

Remark 1. These subspaces that the error vectors belong to are actually subsets of each other, and regarding them as independent subspace one by one is the basis for the new ILC design developed in the next section.

The ILC design objective for problem with non-uniform trial lengths is stated as follows.

Definition 1. The ILC problem is to design an update law

$$u_{k+1} = f(u_k, u_{k-1}, \dots, e_k, e_{k-1}, \dots), \tag{13}$$

to update the input signal for current trial utilizing both trial input and tracking errors that have been already obtained, such that the modified tracking error (12) converges to zero in norm as $k \to \infty$, i.e.

$$\lim_{k\to\infty} \|e_k\| = 0. \tag{14}$$

Note that in ILC it is possible to use information from any previous trials to update the control input to be applied on the next trial. However in this work, only the most common case is considered, i.e., only information from the previous trial is used, which is considered as a kind of efficient strategies for the non-uniform trial length case.[21]

3 | ILC DESIGN USING ALTERNATING PROJECTIONS

In this section, an ILC design for the systems considered is developed by employing alternating projections.

3.1 | Alternating Projections Interpretation

In the case when missing information of the output on a trial is replaced by the corresponding entries in the reference trajectory, the tracking errors for $t \in [N_k + 1,\ N]$ are set as zero. Then, the tracking errors of different trials belong to different subspaces in Hilbert spaces, and there are J subspaces.

In this sense, the ILC design problem formulated in Definition 1 is equivalent to iteratively finding a point in the intersection of the following multiple closed affine subspaces

$$M_j = \left\{(e, u) \in H \,:\, e = F_j(y_d - y), y = Gu + d\right\}, \tag{15}$$

$$M_{J+1} = \{(e, u) \in H \,:\, e = 0\}, \tag{16}$$

where $M_j \in \{M_1, M_2, \dots, M_J\}$ and M_{J+1} respectively represent system dynamics and the tacking objective, and $d \in l_2^{\mathrm{m}}[1,\ N]$. The matrix F_j decides which affine subspace M_j lies in $\{M_1, M_2, \dots, M_J\}$, and is defined as

$$F_j = \begin{bmatrix} I_{N_j} \otimes I_{\mathrm{m}} & \mathbf{0} \\ \mathbf{0} & \mathbf{0}_{(N-N_j)} \otimes I_{\mathrm{m}} \end{bmatrix}, \tag{17}$$

where $N_j = N - j + 1$, for $j \in \{1, 2, \dots, J\}$. In addition, the Hilbert space H is defined as

$$H = l_2^{\ell}[0,\ N-1] \times l_2^{\mathrm{m}}[1,\ N], \tag{18}$$

with the inner product and associated induced norm

$$\langle (e,u), (y,v) \rangle_{\{Q,R\}} = \sum_{i=1}^{N} e^T(i) Q y(i) + \sum_{i=0}^{N-1} u^T(i) R v(i), \tag{19}$$

$$\|(e,u)\|_{\{Q,R\}} = \sqrt{\langle (e,u), (e,u) \rangle_{\{Q,R\}}}. \tag{20}$$

The following assumption is required.

Assumption 1. The multiple affine subspaces M_j and M_{J+1} given by (15) and (16) have intersection region in the Hilbert space H, i.e. $M \cap M_{J+1} \neq \emptyset$, where $M = \bigcap_1^J M_j$.

Assumption 1 guarantees the control objective is achievable, and hence the ILC problem has a solution. Also, due to the existence of the intersection region, there must exist a point $(0, u^*) \in M \cap M_{J+1}$.

Different from successive projections used for the constant trial length case,[28,29] the method of alternating projections used in this paper considers more than two closed sets in the iterative process for the non-uniform trial length case. Therefore, there may be multiple projection orders. To better describe, denote $\{j_k\}_{k\geq 0}$ as a sequence taking values in $\{1, 2, \dots, J\}$, and define a sequence $\{z_k\}_{k\geq 0}$ with

$$z_{k+1} = P_{M_{j_{k+1}}}(z_k),\ k \geq 0, \tag{21}$$

by choosing arbitrary initial point $z_0 \in H$.

Definition 2. The sequence $s = \{j_k\}_{k\geq 0}$ taking i infinitely many times means

$$\delta(s,i) = \sup_k \left[\Delta_{k+1}(i) - \Delta_k(i)\right] < \infty, \tag{22}$$

where $\{\Delta_k(i) \in \mathbb{N}\}_{k\geq 0}$ is an increasing sequence such that $j_{\Delta_k(i)} = i$ with $\Delta_0(i) = 0$.

Taking i infinitely many times requires that the difference in the trial number between the appearance of i at one time and the next is bounded. With Definition 2, the following lemma is required as a basis for the solution of the ILC problem considered in this paper.

Lemma 1. Suppose that M_j, for $j \in \{1, 2, \ldots, J\}$, are closed subspaces in a Hilbert space. If the sequence $s = \left\{j_k\right\}_{k\geq 0}$ takes every value in $\{1, 2, \ldots, J\}$ infinitely many times and there exists a constant S, which is only associated with the sequence s, such that

$$\left\|z_n - z_m\right\|^2 \leq S \sum_{k=m}^{m-1} \left\|z_{k+1} - z_k\right\|^2, \quad n > m \geq 1, \tag{23}$$

then $\left\{z_k\right\}_{k\geq 0}$ converges in norm to the orthogonal projection of z_0 onto $M = \bigcap_1^J M_j$.

Proof. See Appendix A or the work of Sakai[31] for more details. □

Lemma 1 requires that the sequence $s = \left\{j_k\right\}_{k\geq 0}$ takes every value in $i \in \{1, 2, \ldots, J\}$ infinitely many times and hence the possibility of converging by choosing an appropriate sequence $\left\{j_k\right\}_{k\geq 0}$ taking values in $\{1, 2, \ldots, J\}$. However, this is a very strict condition for systems with non-uniform trial lengths, because it is not ensured in practice that actual trial length can take every existent length infinitely many times. Therefore, another assumption is made to relax this condition, while the sequence $\left\{z_k\right\}_{k\geq 0}$ still converges.

Assumption 2. M_1 appears infinitely many times in the process of alternating projections between (15) and (16), i.e.

$$\delta(s, 1) = \sup_k \left[\Delta_{k+1}(1) - \Delta_k(1)\right] < \infty. \tag{24}$$

Note that M_1 represents the system dynamics with $N_k = N$. Therefore, Assumption 2 demands that the case, whose actual length is the desired one, appears infinitely many times and thus the interval between any two sequential trials with desired length is bounded.

Remark 2. This assumption coincides with the persistent full-learning property in the work of Meng and Zhang,[19] where the actual trial can extend to the desired length at least once between any fixed finite interval of successive trials. Similarly, the actual number of $\delta(s, 1)$ has no influence on the convergence result of ILC design because it is only the existence of $\delta(s, 1)$ that matters. Nonetheless, the smaller this value is, the better the learning performance.

Different from Lemma 1, the convergence analysis of alternating projections between (15) and (16) should further consider the affine subspaces of Hilbert space H. In other words, the original point of the Hilbert space H does not naturally belong to the designed affine subspaces. Therefore, a property under a designed projection order is firstly proved to establish the convergence. Denote z^* as a point in the intersection region in Assumption 1, i.e., $z^* = (0, u^*) \in M \cap M_{J+1}$, then the following theorem is given.

Theorem 1. If the projection order satisfies

$$M_{j_k} = \begin{cases} M_j \in \left\{M_1, M_2, \ldots, M_J\right\}, & k \text{ is odd}, \\ M_{J+1}, & k \text{ is even}, \end{cases} \tag{25}$$

then for any $n > m \geq 1$, there exists

$$\langle z_m - z_n, z^* - z_n \rangle \leq 0. \tag{26}$$

Proof. According to Lemma 2 presented in Appendix A, the orthogonal projection operator is idempotent and self-adjoint, then

$$\begin{aligned}\left\langle z - P_{M_j}(z), P_{M_j}(z) - z^* \right\rangle &= \left\langle z - z^*, P_{M_j}(z) - z^* \right\rangle + \left\langle z^* - P_{M_j}(z), P_{M_j}(z) - z^* \right\rangle \\ &= \left\langle z, P_{M_j}(z) \right\rangle - \langle z, z^* \rangle + \left\langle z^*, P_{M_j}(z) \right\rangle - \left\langle P_{M_j}(z), P_{M_j}(z) \right\rangle \\ &= \left\langle z^*, P_{M_j}(z) \right\rangle - \langle z, z^* \rangle = \left\langle P_{M_j}(z^*), z \right\rangle - \langle z, z^* \rangle = 0,\end{aligned} \tag{27}$$

which yields

$$\left\| z - P_{M_j}(z) \right\|^2 = \| z - z^* \|^2 - \left\| P_{M_j}(z) - z^* \right\|^2 - 2 \left\langle z - P_{M_j}(z), P_{M_j}(z) - z^* \right\rangle = \| z - z^* \|^2 - \left\| P_{M_j}(z) - z^* \right\|^2. \tag{28}$$

Reformulating (28) by adding the trial number k yields

$$\left\| z_k - z^* \right\|^2 - \left\| z_{k+1} - z^* \right\|^2 = \left\| z_k - z_{k+1} \right\|^2, \tag{29}$$

then

$$\left\| z_m - z^* \right\|^2 - \left\| z_n - z^* \right\|^2 = \sum_{k=m}^{n-1} \left\| z_{k+1} - z_k \right\|^2, \tag{30}$$

for $n > m \geq 1$. When m is odd and n is even, introduce a scalar λ to establish the relationship between z_m and z_n, then

$$z_n = z_{m-1} + \lambda \left(z_{m+1} - z_{m-1} \right), \ n > m \geq 2. \tag{31}$$

Note that $\left\| z_k - z^* \right\|^2$ monotonically decreases as trial k increases by (29), so $z_n \in M_{J+1}$ should be a point on the line segment with endpoints z_{m+1} and z^* in the Hilbert space H, where $z_{m+1} \in M_{J+1}$. Then,

$$\left\| z_n - z_{m-1} \right\|^2 = \lambda^2 \left\| z_{m+1} - z_{m-1} \right\|^2 \geq \left\| z_{m+1} - z_{m-1} \right\|^2, \tag{32}$$

which yields $\lambda \geq 1$. On the other hand, when $\left\| z_n - z_{m-1} \right\|^2$ converges to 0, it follows that

$$\langle z_n - z_m, z_{m-1} - z_m \rangle = 0, \tag{33}$$

and substituting z_n with (31) yields

$$\lambda = -\frac{\left\| z_m - z_{m-1} \right\|^2}{\left\langle z_{m+1} - z_{m-1}, z_{m-1} - z_m \right\rangle} = \frac{\left\| z_m - z_{m-1} \right\|^2}{\left\langle z_{m+1} - z_{m-1}, z_m - z_{m+1} + z_{m+1} - z_{m-1} \right\rangle} = \frac{\left\| z_m - z_{m-1} \right\|^2}{\left\| z_{m+1} - z_{m-1} \right\|^2}, \tag{34}$$

since $\left\langle z_{m+1} - z_{m-1}, z_m - z_{m+1} \right\rangle = 0$, which has similar proof with (27). Then, it follows that

$$1 \leq \lambda \leq \frac{\left\| z_m - z_{m-1} \right\|^2}{\left\| z_{m+1} - z_{m-1} \right\|^2}. \tag{35}$$

Moreover, reformulating $\langle z_m - z_n, z^* - z_n \rangle$ yields

$$\begin{aligned}
\langle z_m - z_n, z^* - z_n \rangle &= \left\langle \left(z_m - z_{m+1}\right) + \left(z_{m+1} - z_n\right), z^* - z_{m-1} - \lambda\left(z_{m+1} - z_{m-1}\right)\right\rangle \\
&= \left\langle z_m - z_{m+1}, (1-\lambda)\left(z^* - z_{m-1}\right)\right\rangle + \left\langle z_m - z_{m+1}, \lambda\left(z^* - z_{m+1}\right)\right\rangle \\
&+ \left\langle (1-\lambda)\left(z_{m+1} - z_{m-1}\right), \left(z^* - z_{m-1}\right) - \lambda\left(z_{m+1} - z_{m-1}\right)\right\rangle \\
&= \left\langle z_m - z_{m-1} + z_{m-1}, (1-\lambda)\left(z^* - z_{m-1}\right)\right\rangle - (1-\lambda)\left\langle z_{m+1}, z^* - z_{m-1}\right\rangle \\
&+ (1-\lambda)\left\langle z_{m+1}, z^* - z_{m-1}\right\rangle - (1-\lambda)\langle z_{m-1}, z^* - z_{m-1}\rangle - (1-\lambda)\lambda\left\|z_{m+1} - z_{m-1}\right\|^2 \\
&= (1-\lambda)\left(\langle z_m - z_{m-1}, z^* - z_{m-1}\rangle - \lambda\left\|z_{m+1} - z_{m-1}\right\|^2\right),
\end{aligned} \tag{36}$$

by $\left\langle z_m - z_{m+1}, z_{m+1} - z^*\right\rangle = 0$ and $\langle z_m - z_{m-1}, z_m - z^*\rangle = 0$.

Except for $\lambda = 1$, substituting λ with $\frac{\|z_m - z_{m-1}\|^2}{\|z_{m+1} - z_{m-1}\|^2}$ also yields $\langle z_m - z_n, z^* - z_n\rangle = 0$, because in (36), there exists

$$\begin{aligned}
\langle z_m - z_{m-1}, z^* - z_{m-1}\rangle - \lambda\left\|z_{m+1} - z_{m-1}\right\|^2 &= \langle z_m - z_{m-1}, z^* - z_{m-1}\rangle - \langle z_m - z_{m-1}, z_m - z_{m-1}\rangle \\
&= \langle z_m - z_{m-1}, z^* - z_m\rangle = 0.
\end{aligned} \tag{37}$$

Note that (36) is eventually transformed into a quadratic function with respect to λ with a positive quadratic coefficient, therefore $\langle z_m - z_n, z^* - z_n\rangle \leq 0$ by (35) for $n > m \geq 2$. For $n > m = 1$, the utilization of z_2, z_1 and $P^{-1}_{M_{J+1}}\left(z_1\right)$ yields the same result, where $P^{-1}_{M_{J+1}}\left(z_1\right)$, belonging to M_{J+1}, represents the original orthogonal projection point of z_1.

When m is even and n is odd, by employing two auxiliary points, i.e., $P_{M_{j_n}}\left(z_m\right)$ and $P^{-1}_{M_{j_n}}\left(z_m\right)$, the same result with the case where n is even and m is odd can be established. When both m and n are odd or even, we can also achieve the result (26) since $\left\|z_k - z^*\right\|^2$ monotonically decreases as the trial number k increases, even though the two affine subspaces may not be same when both m and n are odd. Finally, the proof is complete. □

According to Theorem 1, the convergence result can now be established in the next theorem.

Theorem 2. The sequence $\left\{z_k\right\}_{k\geq 0}$ converges in norm to the orthogonal projection of z_0 onto $M \cap M_{J+1}$ under the projection order (25).

Proof. It follows from the result (26) in Theorem 1 that

$$\left\|z_n - z_m\right\|^2 = \left\|z_m - z^*\right\|^2 - \left\|z_n - z^*\right\|^2 + 2\langle z_m - z_n, z^* - z_n\rangle \leq \left\|z_m - z^*\right\|^2 - \left\|z_n - z^*\right\|^2,\ n > m \geq 1, \tag{38}$$

then combined with (30), it follows that

$$\left\|z_n - z_m\right\|^2 \leq \sum_{k=m}^{n-1}\left\|z_{k+1} - z_k\right\|^2,\ n > m \geq 1. \tag{39}$$

Recall that (39) is the condition ensuring that the sequence $\{z_k\}_{k\geq 0}$ converges in norm with $S = 1$ as Lemma 1 states, while the difference lies in that all M_j, $j \in \{1, 2, \dots, J\}$, are affine subspaces for Theorem 2.

Since $\|z_k - z^*\|^2$ monotonically decreases as k increases and bounded below by 0 according to (29), so there exists a constant $\gamma > 0$ such that $\lim_{k\to\infty} \|z_k - z^*\|^2 = \gamma$. Moreover, given a constant $\epsilon > 0$, there exists $k \in \mathbb{N}$ such that $0 \leq \|z_m - z^*\|^2 - \gamma < \epsilon/2$ whenever $m \geq k$ and it also works when it comes to $n \geq k$. Then, it follows from (38) that

$$\|z_n - z_m\|^2 \leq \|z_m - z^*\|^2 - \gamma + \gamma - \|z_n - z^*\|^2 < \epsilon/2 + \epsilon/2 = \epsilon, \tag{40}$$

which yields that $\{z_k\}_{k\geq 0}$ converges in norm to a point according to the completeness of Hilbert spaces. For brevity, the convergent point is denoted by z_∞.

Note that M_1 appears infinitely many times as stated in Assumption 2, so there is a convergent sub-sequence $\{z_{\Delta_k(1)}\}_{k\geq 0}$ such that each $z_{\Delta_k(1)} \in M_1$. Therefore, there exists $\langle z_{\Delta_k(1)}, z' \rangle = 0$ for every point $z' \in M_1^\perp$, which gives rise to

$$\langle z_\infty, z' \rangle = \left\langle \lim_{k\to\infty} z_{\Delta_k(J+1)}, z' \right\rangle = \lim_{k\to\infty} \langle z_{\Delta_k(J+1)}, z' \rangle = 0, \tag{41}$$

and hence $z_\infty \in M_1$. Since M_{J+1} also appears infinitely many times under the designed projection order (25) with $\delta(s, J+1) = 2$, so $z_\infty \in M_{J+1}$. Then, $z_\infty = (0, u_\infty) \in M_1 \cap M_{J+1}$, where u_∞ is the convergent control input and

$$e = 0 = F_1 (y_d - Gu_\infty - d) = (y_d - Gu_\infty - d), \tag{42}$$

since $F_1 = I_N$ by (17). When it comes to M_j, it follows that

$$e = F_j (y_d - Gu_\infty - d) = 0, \tag{43}$$

for each $j \in \{2, 3, \dots, J\}$. Then, $z_\infty = (0, u_\infty) \in M_j$ for every $j \in \{2, 3, \dots, J\}$ and hence $z_\infty \in M \cap M_{J+1}$ since $M = \bigcap_1^J M_j$.

Furthermore, considering the following subspaces to project on in the Hilbert space H yields $z_k - P_{M_{j_{k+1}}}(z_k) \in M_{j_{k+1}}^\perp$. Note also that $z^* \in M \cap M_{J+1}$ and thus $z^* \in M_{j_{k+1}}$, then

$$\langle z_k - z_{k+1}, z^* \rangle = \left\langle z_k - P_{M_{j_{k+1}}}(z_k), z^* \right\rangle = 0, \tag{44}$$

which yields

$$\langle z_0 - z_\infty, z^* \rangle = \lim_{k\to\infty} \langle z_0 - z_{k+1}, z^* \rangle = \lim_{k\to\infty} (\langle z_0 - z_1, z^* \rangle + \cdots + \langle z_k - z_{k+1}, z^* \rangle) = 0. \tag{45}$$

Hence, $z_0 - z_\infty \in (M \cap M_{J+1})^\perp$. According to the projection theorem in Hilbert spaces, z_∞ is the orthogonal projection of z_0 onto $M \cap M_{J+1}$ by $z_0 = z_\infty + (z_0 - z_\infty)$ and the proof is complete. □

The difference between z^* and z_∞ lies in that, z^* can be any point that exists in $M \cap M_{J+1}$ according to Assumption 1, while z_∞ is the convergent point of the sequence $\{z_k\}_{k\geq 0}$ under the designed projection order (25). Also, z_∞ belongs to $M \cap M_{J+1}$ by Theorem 2. Next, the result of Theorem 2 is used to obtain an ILC law design.

3.2 | Optimal ILC Design and Convergence Analysis

To design an optimal ILC update law, a cost function should be designed for each trial to reduce the tracking error or other targets. According to the alternating projection interpretation, the distance $\|z_{k+1} - z_k\|$ in Hilbert space H is going to be reduced. Therefore, according to the inner product and associated induced norm (19) and (20), the cost function can be taken as

$$J\left(u_{k+1}\right) = \|z_{k+1} - z_k\|^2 = \|e_{k+1}\|_Q^2 + \|u_{k+1} - u_k\|_R^2 . \tag{46}$$

Moreover, the norm optimal ILC update law as used in the work of Amann et al[32] can be employed to deal with (46), i.e.

$$u_{k+1} = u_k + G^* e_{k+1}, \tag{47}$$

where G^* denotes the adjoint operator of G in Hilbert space and $I = I_N \otimes I_{\mathrm{m}}$.

Remark 3. Due to the property of adjoint operator, the form of norm optimal ILC update law (47) is not causal, and is not implementable. It will be shown later in this paper that (47) can be reformulated to enable implementation as a simple feedforward or causal feedback plus feedforward structure.

The next result establishes that the ILC update law (47) solves the problem given in Definition 1.

Proposition 1. The input sequence $\{u_k\}_{k\geq 0}$ generated by update law (47) iteratively solves the ILC problem with non-uniform trial lengths given in Definition 1.

Proof. With the multiple affine subspaces defined in (15) and (16), the ILC problem with non-uniform trial lengths is transformed into the projection problem onto M_{j_k}, where alternating sequence $\{M_{j_k}\}_{k\geq 1}$ takes values in the order of $\{M_j, M_{J+1}, M_j, M_{J+1}, \dots\}$ under (25). In this sense, let $\tilde{z} = (\tilde{e}, \tilde{u}) \in M_j$ where $j \in \{1, 2, \dots, J\}$ and $z = (0, u) \in M_{J+1}$. Hence, for $j \in \{1, 2, \dots, J\}$,

$$\begin{aligned} P_{M_j}(z) &= \arg\min_{\hat{z}\in M_j} \|\hat{z} - z\|_H^2 = \arg\min_{(\hat{e},\hat{u})\in M_j} \|(\hat{e}, \hat{u}) - (0, u)\|_{\{Q,R\}}^2 \\ &= \arg\min_{(\hat{e},\hat{u})\in M_j} \left\{ \|\hat{e} - 0\|_Q^2 + \|\hat{u} - u\|_R^2 \right\} = \arg\min_{\hat{u}} \left\{ \|\hat{e}\|_Q^2 + \|\hat{u} - u\|_R^2 \right\}, \end{aligned} \tag{48}$$

which is an optimization problem and is solved by update law (47). Similarly,

$$P_{M_{J+1}}(\tilde{z}) = \arg\min_{\hat{z}\in M_{J+1}} \|\hat{z} - \tilde{z}\|_H^2 = \arg\min_{(0,\hat{u})\in M_{J+1}} \|(0, \hat{u}) - (\tilde{e}, \tilde{u})\|_{\{Q,R\}}^2 = \arg\min_{(0,\hat{u})\in M_{J+1}} \left\{ \|0 - \tilde{e}\|_Q^2 + \|\hat{u} - \tilde{u}\|_R^2 \right\}, \tag{49}$$

whose solution is $\hat{u} = \tilde{u}$ when $(0, \hat{u}) \in M_{J+1}$. In this sense, using (47) to solve (48) and (49) repeatedly means alternating projections under the designed order (25). Therefore, the resulting sequence $\{u_k\}_{k\geq 0}$ generated by update law (47) solves the ILC problem in Definition 1. □

The convergence of the new design will now be analyzed with the alternating projections of Theorem 2.

According to Assumption 1, there exists a point z^* belonging to the intersection region of the multiple subspaces M_j and M_{J+1}, which means systems with non-uniform trial lengths can eventually operate with zero tracking error. This convergence property is established by the following theorem.

Theorem 3. Given system (1) with initial input u_0, if Assumption 1 holds, application of the optimal ILC update law (47) results in

$$\lim_{k\to\infty} u_k = u_\infty, \quad \lim_{k\to\infty} \|e_k\| = 0. \tag{50}$$

Proof. Since the sequence $\{z_k\}_{k\geq 0}$ converges in norm to $z_\infty = (0, u_\infty)$ under the projection order of (25) by Theorem 2, the distance between each two projections converges to 0. Hence, given the cost function (46), it follows that

$$\lim_{k\to\infty} \left\{ \|0 - e_k\|_Q^2 + \|u_\infty - u_k\|_R^2 \right\} = 0, \tag{51}$$

which establishes (50) by Assumption 1. □

Remark 4. Although the norm optimal ILC update law (47) is applied, monotonic convergence property of the modified tracking error in norm cannot be achieved in general because the actual lengths are not identical. However, when k is even, there exists $\langle z_k - z_{k+2}, z_{k+2} - z_{k+1} \rangle = 0$, which can be proved by (27) similarly, and it follows that

$$\|z_k - z_{k+1}\|^2 = \|z_k - z_{k+2}\|^2 + \|z_{k+2} - z_{k+1}\|^2 + 2\langle z_k - z_{k+2}, z_{k+2} - z_{k+1} \rangle \geq \|z_{k+1} - z_{k+2}\|^2, \tag{52}$$

which shows that monotonic performance under (25) in Theorem 2 is possible. When k is odd, however, (52) does not always hold because both z_k and z_{k+2} are not always located in the same affine subspace defined in (15), i.e., each two actual trial lengths of the ILC process are not always identical.

In Theorem 3, the convergence of the ILC design for systems with non-uniform trial lengths is proved under the alternating projection framework. Next, it is shown that the ILC law can be reformulated to allow implementation.

3.3 | Causal Feedback Plus Feedforward Implementation

The following steps compose a causal implementation procedure for the ILC law considered in this paper.

Step 1 Input the system dynamics (1), initial input u_0, positive definite matrices Q and R and stopping criterion value $\sigma > 0$, and set $k = 0$;

Step 2 Calculate the state feedback matrices $K(t)$ for $t \in [0,\ N-1]$ using the Riccati equation

$$K(t) = A^T K(t+1)\left[I_n + BR^{-1}B^T K(t+1)\right]^{-1} A + C^T QC, \tag{53}$$

with the boundary condition $K(N) = 0$;

Step 3 Set $k = k + 1$, then calculate feedforward terms $\xi_{k+1}(t)$ for $t \in [0,\ N-1]$ by the difference equation

$$\xi_{k+1}(t) = \left[I_n + K(t)BR^{-1}B^T\right]^{-1}\left[A^T \xi_{k+1}(t+1) + C^T Q e_k(t+1)\right], \tag{54}$$

with the boundary condition $\xi_{k+1}(N) = 0$;

Step 4 Calculate the control input $u_{k+1}(t)$ until $t = N_{k+1} - 1$ by

$$u_{k+1}(t) = u_k(t) + R^{-1}B^T p_{k+1}(t), \tag{55}$$

with

$$p_{k+1}(t) = -K(t)\left[I_n + BR^{-1}B^T K(t)\right]^{-1} A \times \left[x_{k+1}(t) - x_k(t)\right] + \xi_{k+1}(t), \tag{56}$$

where $p_{k+1}(t)$ is a defined costate vector;

Step 5 Set $u_{k+1}(t) = u_k(t)$ for $t \in \left[N_{k+1},\ N-1\right]$;

Step 6 If $\left\|e_{k+1}\right\| < \sigma$, finish the procedure, otherwise return to Step 3.

Remark 5. Steps 1-6 above compose a practical implementation for systems with non-uniform trial lengths, where there exists an extended setting on the input signal when the current trial is ended prematurely. In this case, this procedure can handle systems with non-uniform trial lengths for some complex situations in practice by adjusting to practical requirements. For instance, when $t \in \left[N_{k+1},\ N-1\right]$, set $u_{k+1}(t)$ as some actual achievable values for safety reasons or just zero to avoid wasted computational effort.

The next result formally establishes this implementation procedure.

Proposition 2. The norm optimal ILC update law (47) for systems with non-uniform trial lengths can be implemented using the feedback plus feedforward structure given as Steps 1-6.

Proof. See Appendix B for the details. □

4 | EXTENSION TO INPUT CONSTRAINTS

When considering constraints on the input signal, M_{J+1} may not be a closed subspace but still a closed set. Furthermore, the constrained set is usually convex in practice. Therefore, the convex constraint on the input signal can be embedded into the tracking objective, i.e.

$$M_j = \left\{(e,u) \in H \,:\, e = F_j\left(y_d - y\right), y = Gu + d\right\} \tag{57}$$

$$M_{J+1} = \{(e,u) \in H \,:\, e = 0, u \in \Omega\}, \tag{58}$$

where Ω is a closed convex set that represents the input constraints and also $M_j \in \left\{M_1, M_2, \dots, M_J\right\}$.

Remark 6. The reason why the input constraints is embedded into M_{J+1}, instead of M_j for $j \in \{1, 2, \dots, J\}$, is that $P_{M_{J+1}}(\tilde{z})$ is equivalent to $P_\Omega(\tilde{u})$ by (49) when finding the projection point on M_{J+1} with input constraints. On the contrary, if embedding the input constraints into M_j, a complex constrained optimization problem is to be solved. See the work of Chu and Owens[28] for a detailed discussion.

Note that when applying alternating projections between (57) and (58), the projection sequence can still be proved to converge in norm if Assumption 1 still holds. Nevertheless, the convergent point may not be the orthogonal projection of initial point onto the intersection region. Although faster convergence speed occurs under convergence to the projection of initial point, this property is still ensured when the convergent point belongs to the region. In this case, a theorem for ILC design problem with non-uniform trial lengths under input constraints is established next.

Theorem 4. If M_{J+1} is a closed convex set and Assumption 1 still holds, the sequence $\left\{z_k\right\}_{k\geq 0}$ converges in norm to a point that belongs to $M \cap M_{J+1}$ under the projection order of (25).

Proof. Due to the convexity of M_{J+1}, there exists

$$\left\langle z_k - P_{M_{J+1}}\left(z_k\right), P_{M_{J+1}}\left(z_k\right) - z\right\rangle \geq 0, \tag{59}$$

for any $z \in M_{J+1}$. In particular, when k is even, there exists $\left\langle z_{k+1} - z_{k+2}, z_{k+2} - z_k\right\rangle \geq 0$, and hence

$$\left\|z_{k+1} - z_k\right\|^2 = \left\|z_{k+1} - z_{k+2}\right\|^2 + \left\|z_{k+2} - z_k\right\|^2 + 2\left\langle z_{k+1} - z_{k+2}, z_{k+2} - z_k\right\rangle \geq \left\|z_{k+1} - z_{k+2}\right\|^2. \tag{60}$$

When k is odd, (60) may not always hold because it comes to the affine subspaces under the projection order (25), i.e., M_j, instead of just convex sets. Nonetheless, it similarly follows from (27) that

$$\left\langle z_k - P_{M_j}\left(z_k\right), P_{M_j}\left(z_k\right) - z'\right\rangle = 0, \tag{61}$$

for any $z' \in M_j$, where $j \in \{1, 2, \ldots, J\}$. Therefore, for $z^* \in M \cap M_{J+1}$ and all k, there exists $\left\langle z_k - z_{k+1}, z_{k+1} - z^* \right\rangle \geq 0$, which yields

$$\left\|z_k - z^*\right\|^2 = \left\|z_k - z_{k+1}\right\|^2 + \left\|z_{k+1} - z^*\right\|^2 + 2\left\langle z_k - z_{k+1}, z_{k+1} - z^* \right\rangle \geq \left\|z_k - z_{k+1}\right\|^2 + \left\|z_{k+1} - z^*\right\|^2. \tag{62}$$

Furthermore,

$$\left\|z_0 - z^*\right\|^2 \geq \left\|z_k - z^*\right\|^2 + \sum_{i=0}^{k-1} \left\|z_i - z_{i+1}\right\|^2, \tag{63}$$

and when $k \to \infty$,

$$\infty > \left\|z_0 - z^*\right\|^2 \geq \sum_{i=0}^{\infty} \left\|z_i - z_{i+1}\right\|^2. \tag{64}$$

Therefore, when $k \to \infty$, it follows that

$$\inf_{z \in M_{j_{k+1}}} \left\|z_k - z\right\| \to 0, \tag{65}$$

and hence the sequence $\left\{z_k\right\}_{k \geq 0}$ converges in norm to a point belonging to $M \cap M_{J+1}$ in the defined finite-dimensional Hilbert space H. □

Using Theorem 4, an optimal ILC update law for systems with non-uniform trial lengths under input constraints is next designed and it consists of two parts. The first part is to find the optimal solution in the absence of the input constraints, i.e.,

$$\tilde{u}_{k+1} = u_k + G^* e_{k+1}, \tag{66}$$

which is consistent with (47) and can be implemented using Steps 1-6 given in the previous section. The second part is to project the optimal solution onto the constraint set Ω, i.e.,

$$u_{k+1} = \arg\min_{u \in \Omega} \left\{ \left\|u - \tilde{u}_{k+1}\right\|_R^2 \right\}, \tag{67}$$

which could be implemented by setting constraints on $u_k(t)$ in Steps 1-6.

The next result shows that the ILC problem in Definition 1 can be solved by (66) and (67) in the presence of input constraints.

Proposition 3. The input sequence $\left\{u_k\right\}_{k \geq 0}$ generated by update law (66) and (67) iteratively solves the ILC problem with non-uniform trial lengths in Definition 1 under input constraints.

Proof. With the multiple closed sets defined in (57) and (58), the ILC problem can be still transformed into the projection problem onto M_{j_k} with the order of $\left\{M_j, M_{J+1}, M_j, M_{J+1}, \ldots\right\}$. Similar to the proof of Proposition 1, it follows that

$$P_{M_j}(z) = \arg\min_{\hat{u}} \left\{ \|\hat{e}\|_Q^2 + \|\hat{u} - u\|_R^2 \right\}, \tag{68}$$

for $j \in \{1, 2, \ldots, J\}$, which can be solved by (66). Projecting on M_{J+1} gives

$$P_{M_{J+1}}(\tilde{z}) = \underset{(0,\hat{u})\in M_{J+1}}{\arg\min} \left\{ \|0 - \tilde{e}\|_Q^2 + \|\hat{u} - \tilde{u}\|_R^2 \right\} = \underset{\hat{u}\in\Omega}{\arg\min} \left\{ \|\hat{u} - \tilde{u}\|_R^2 \right\}, \tag{69}$$

which can be solved by (67). Then, the sequence $\{u_k\}_{k\geq 0}$ generated by the update law (66) and (67) solves the ILC problem in Definition 1 in this case. □

Although M_{J+1} becomes a closed convex set for the problem considered, the convergence of alternating projections, in the order of (25), can be still guaranteed under Assumption 1. Hence the convergence property of ILC update law (66) and (67) can be established as the following theorem.

Theorem 5. In the presence of input constraints, consider a system described by (1) with initial input $u_0 \in \Omega$. If Assumption 1 still holds and both optimal ILC update law (66) and (67) are applied, then

$$\lim_{k\to\infty} u_k = u^*, \quad \lim_{k\to\infty} \|e_k\| = 0. \tag{70}$$

Proof. By Theorem 4 and Proposition 3, although M_{J+1} is a closed convex set, the sequence $\{z_k\}_{k\geq 0}$ converges in norm to a point that belongs to $M \cap M_{J+1}$ when applying the ILC update law (66) and (67). Therefore, the distance between each two projections converges to 0, and hence

$$\lim_{k\to\infty} \left\{ \|0 - e_k\|_Q^2 + \|u^* - u_k\|_R^2 \right\} = 0. \tag{71}$$

By Assumption 1, the convergence (70) is established. □

In the presence of input constraints, the specific monotonic performance of Remark 4 for the unconstrained case still exists. Next, a numerical case study is given to illustrate the new results in this paper.

5 | NUMERICAL CASE STUDY

In this section, a coarse-fine stage is employed to verify the effectiveness of the new design. The coarse-fine stage uses multiple actuators, where coarse and fine actuators are respectively in charge of long and short range positioning. High-precision positioning is usually required in many of its practical applications, including some that perform repeating tasks. Therefore, ILC is usually applied to such systems for high tracking performance.[33] However, varying trial lengths may happen during the iterative learning process because of some unexpected obstacles in the path or other kinds of output constraints, which gives rise to the non-uniform trial length case. In this simulation, the ILC control problem with non-uniform trial lengths in the control of a coarse-fine stage is considered.

5.1 | Modeling and Design

The employed coarse-fine stage consists of two parts[33]: the coarse stage employs a rotary motor to drive a linear ball-screw stage, and the fine stage is driven by a voice coil actuator. Both outputs of the two stages are the position relative to the ground. Denote the superscript $(\cdot)$ as the components of a vector, then inputs and outputs of the coarse and fine stages can be denoted by $u^{(1)}$, $y^{(1)}$, $u^{(2)}$ and $y^{(2)}$, respectively. The transfer functions from $u^{(1)}$ to $y^{(1)}$, $u^{(1)}$ to $y^{(2)}$, $u^{(2)}$ to $y^{(1)}$ and $u^{(2)}$ to $y^{(2)}$ are as follows:

$$P_{11}(s) = \frac{m_2 s^2 + cs + k}{D(s)},\ P_{12}(s) = \frac{m_2 s^2}{D(s)},\ P_{21}(s) = \frac{cs + k}{D(s)},\ P_{22}(s) = \frac{m_1 s^2 + bs}{D(s)},$$
$$D(s) = m_1 m_2 s^4 + \left(bm_2 + cm_1 + cm_2\right) s^3 + \left(bc + km_1 + km_2\right) s^2 + bks, \tag{72}$$

where m_1 and m_2 respectively denote the masses of the coarse and fine stages, k and c respectively denote the stiffness and viscous damping coefficient between the two stages, and b denotes the coefficient of viscous damping between the ground and the coarse stage. For more modeling details, please refer to the work of Yoon et al.[33] The model parameters are as follows:

$$m_1 = 39.3\ \text{kg},\ \text{m}_2 = 0.5\ \text{kg},\ \text{b} = 60\ \text{Ns/m},\ k = 10^5\ \text{N/m},\ \text{c} = 45\ \text{Ns/m}. \tag{73}$$

For the non-uniform trial length situation, set the maximum and minimum tracking time of the coarse-fine stage as 2s and 1.8s respectively, which means that the actual length N_k varies from $N_m = 180$ to $N = 200$ with sample time $T_s = 0.01$s. Note that the new design requires no settings on the distribution of N_k, and a discrete uniform distribution is employed here for simplicity. Without loss of generality, set $x_k(0) = [\underbrace{0\ \cdots\ 0}_{\text{n}}]^T$ and $u_0(t) = [0,\ 0]^T$, for $t \in [0,\ N-1]$. The desired trajectory of the positioning is taken as

$$y_d^{(1)}(t) = y_d^{(2)}(t) = 1.6t^2 \left[1 + \cos\left(\frac{\pi t}{4} - \pi\right)\right], \tag{74}$$

which means the outputs of both coarse and fine stages follow the same paths and the initial positions of the two stages are identical.

5.2 | Simulation Results

The simulation is implemented in MATLAB R2020a. The weighting matrices are firstly selected as $Q = 10000 I_\text{m}$ and $R = 0.001 I_\ell$, respectively. The design is for a total of 20 trials, and the 2nd, 4th and 20th output profiles are shown in Fig. 1. The output of 20th trial can track the desired trajectory for $1 \leq t \leq N_{20}$ and the output of the first few trials are also plotted with their actual trial lengths in Fig. 1. In particular, the outputs of the 4th trial have worse tracking performance during the interval $\left[N_k, N\right]$ and this also possibly occurs at the 20th trial especially when the number of desired length occurs less, which coincides with comments in Remark 2. The variation of the trial lengths along the trial is given in Fig. 2.

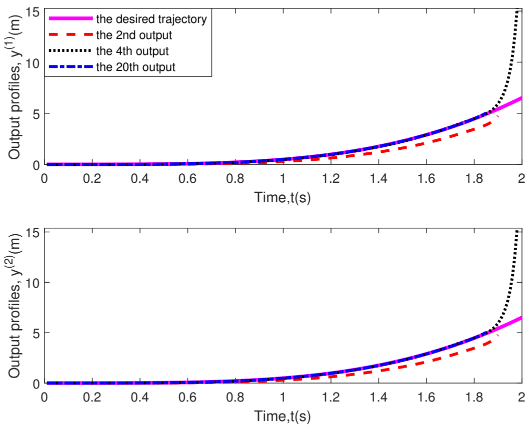

FIGURE 1 The 2nd, 4th and 20th output profiles under the new ILC design with the desired trajectory.

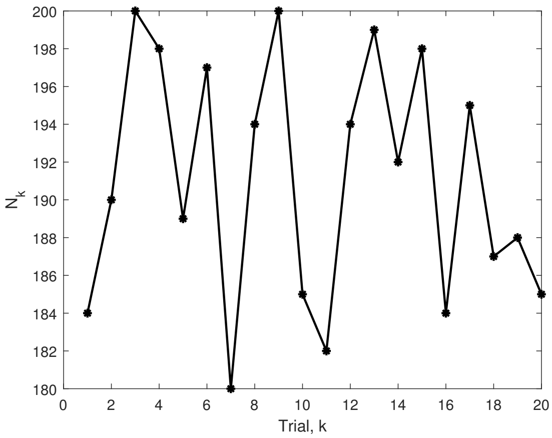

FIGURE 2 The variation of the actual trial lengths.

The tracking errors in 2-norm along the trial are plotted in Fig. 3, which confirms that the tracking errors can converge asymptotically to zero. For comparisons, the ILC method based on an iterative average operator [14] is employed with almost best tuned learning gain $5I_m$, whose tracking errors in 2-norm are also plotted in Fig. 3. Moreover, the P-type ILC method with Arimoto-like gain [17] is also simulated, whose learning parameters are tuned to $20I_m$. The 2-norm of tracking errors in logarithmic coordinates is also embedded in Fig. 3. It is shown that the new ILC design converges faster than these two classic methods. The cost function of the new ILC design defined in (46) is given in Fig. 4. The monotonic convergence cannot be obtained because of the non-uniform trial length case, which is consistent with the discussion in Remark 4.

Different choices of the weighting matrices Q and R can result in different convergence performance of the new ILC design. Fig. 5 gives the results of different Q and R, where increasing Q or decreasing R will result in faster convergence speed. From an intuitive point of view, both Q and R can decide the angle between sets defined in (15) and (16) in Hilbert space H. Changes

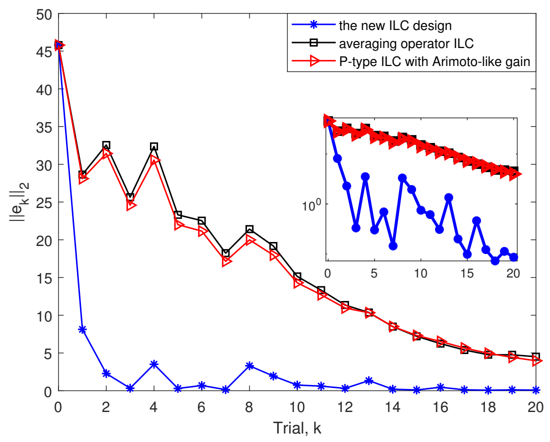

FIGURE 3 The tracking errors in 2-norm of the new ILC design and classic ILC methods along the trial.

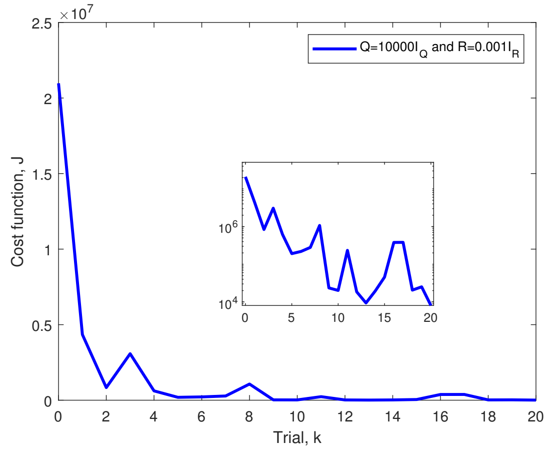

FIGURE 4 The cost function of the new ILC design along the trial.

of the angle will fundamentally affect the results of the convergent sequence $\{z_k\}_{k\geq 0}$ and eventually affect the performance of the new ILC design.

In addition, to check the constraint handling capability, the new ILC design is applied under input constraints. Fig. 6 and Fig. 7 respectively present the 2nd, 4th and 20th output and input profiles with the input constraint $[-3000\text{N}, 3000\text{N}]$. The actual output can still track the desired trajectory under input constraints after certain trials. The cost function of the new ILC design under the saturation constraint is shown in Fig. 8. It is still convergent and there is less fluctuation along the trials.

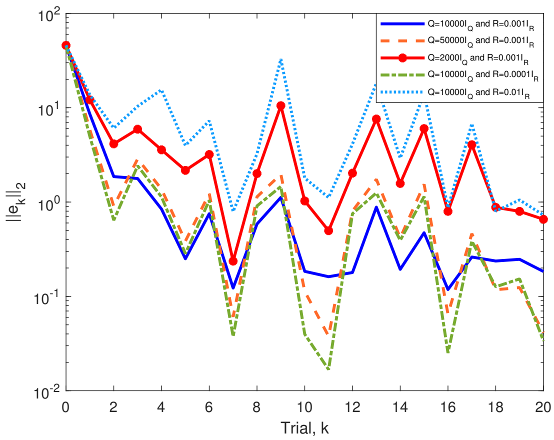

FIGURE 5 The tracking errors in 2-norm of different choices for weighting matrices Q and R.

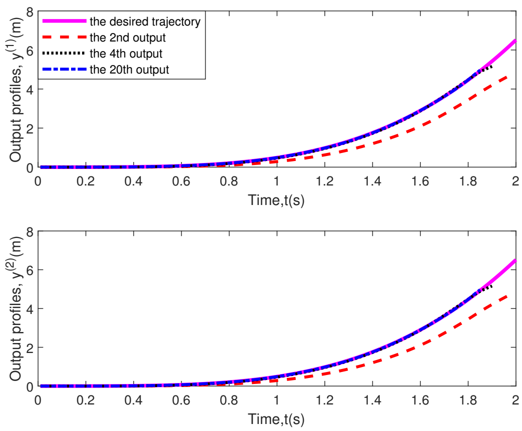

FIGURE 6 The 2nd, 4th and 20th output profiles under the new ILC design with input constraints.

6 | CONCLUSION AND FUTURE WORK

In this paper, a novel alternating projection framework has been developed for ILC design with non-uniform trial lengths. The causal feedback plus feedforward structure of the uniform norm optimal ILC was modified to give an implementation for discrete-time systems with non-uniform trial lengths. Furthermore, it has been shown that alternating projections for analysis extends to allow input constraints without the need to solve complex optimization problems. Moreover, the convergence properties of the new ILC design were analyzed theoretically. Finally, a numerical simulation based on the model of a coarse-fine stage has been given to demonstrate the effectiveness of the new design for discrete-time systems, including a comparison with two alternative designs, namely, the iterative average ILC and P-type ILC with Arimoto-like gain.

For future work, the ILC design for continuous-time systems with non-uniform trial lengths, where there exists infinite number of sets, will be studied. Furthermore, the new design will be implemented in practice to determine its experimental performance.

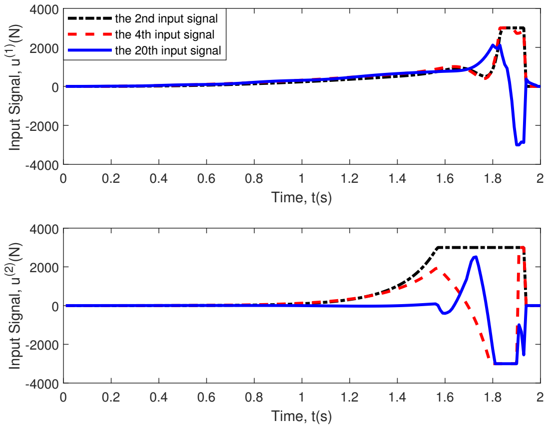

FIGURE 7 The 2nd, 4th and 20th input profiles of the new ILC design with input constraints.

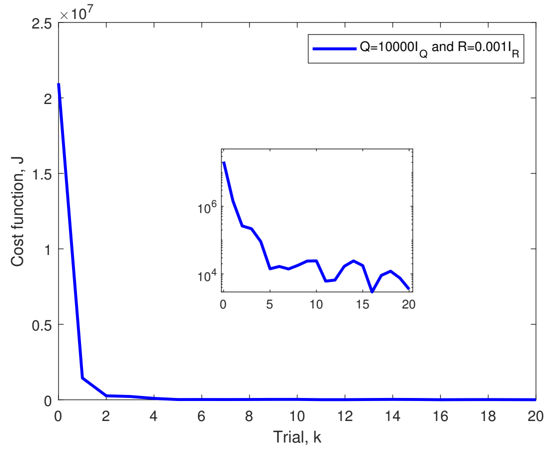

FIGURE 8 The cost function of the new ILC design under input constraints.

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□

APPENDIX

A PROOF OF LEMMA 1

Before proving Lemma 1, a technical lemma is firstly introduced.

Lemma 2. The projection operator P is idempotent and self-adjoint.

Proof. According to the projection theorem in Hilbert spaces, given a Hilbert space H and a subspace $Z \subset H$, each $z \in H$ can be written uniquely as $z = z_1 + z_2$, where $z_1 \in Z$ and $z_2 \in Z^{\perp}$. Then

$$P_Z^2(z) = P_Z\left(P_Z\left(z_1 + z_2\right)\right) = P_Z\left(z_1\right) = z_1 = P_Z(z), \tag{A1}$$

for the proof of idempotency. Given another $z' \in H$, there exist unique $z'_1 \in Z$ and $z'_2 \in Z^{\perp}$, then

$$\left\langle P_Z(z), z' \right\rangle = \left\langle z_1, z'_1 + z'_2 \right\rangle = \left\langle z_1 + z_2, z'_1 \right\rangle = \left\langle z, P_Z\left(z'\right) \right\rangle, \tag{A2}$$

for the proof of self-adjointness. □

Proof of Lemma 1. According to Lemma 2, it follows that

$$\left\langle P_{M_j}(z), z - P_{M_j}(z) \right\rangle = \left\langle P_{M_j}(z), z \right\rangle - \left\langle P_{M_j}(z), P_{M_j}(z) \right\rangle = \left\langle P_{M_j}(z), z \right\rangle - \left\langle P_{M_j}\left(P_{M_j}(z)\right), z \right\rangle = 0, \tag{A3}$$

which yields $z - P_{M_j}(z) \perp P_{M_j}(z)$ and $z_k - z_{k+1} \perp z_{k+1}$ by adding the trial number k. Then, it follows from (A3) that $\left\|z_k\right\|^2 = \left\|z_{k+1}\right\|^2 + \left\|z_k - z_{k+1}\right\|^2$, and doing recursion yields

$$\left\|z_m\right\|^2 = \left\|z_n\right\|^2 + \sum_{k=m}^{n-1} \left\|z_{k+1} - z_k\right\|^2. \tag{A4}$$

Substituting (A4) into the given condition (23) yields $\left\|z_n - z_m\right\|^2 \le S\left(\left\|z_m\right\|^2 - \left\|z_n\right\|^2\right)$. Also, it follows from (A4) that $\left\|z_k\right\|^2$ is monotonically decreasing and bounded below by 0, so there exists a constant $\alpha \ge 0$ such that $\lim_{k\to\infty} \left\|z_k\right\|^2 = \alpha$. Furthermore, given $\varepsilon > 0$, there exists $k \in \mathbb{N}$ such that $0 \le \left\|z_n\right\|^2 - \alpha < \varepsilon/2S$ whenever $n \ge k$. Therefore,

$$\left\|z_n - z_m\right\|^2 \le S\left(\left\|z_m\right\|^2 - \alpha + \alpha - \left\|z_n\right\|^2\right) < S \cdot \varepsilon/2S + S \cdot \varepsilon/2S = \varepsilon, \tag{A5}$$

and since the sequence $\left\{z_k\right\}_{k\ge0}$ is a Cauchy sequence in Hilbert spaces, $\left\{z_k\right\}_{k\ge0}$ converges in norm to a point, which is denoted by z_∞.

Moreover, since $\left\{j_k\right\}_{k\ge0}$ takes every value in $\{1, 2, \dots, J\}$ infinitely many times, so there is a sub-sequence $\left\{z_{\Delta_k(i)}\right\}_{k\ge0}$ such that each $z_{\Delta_k(i)} \in M_j$. Then, there exists $\left\langle z_{\Delta_k(i)}, z' \right\rangle = 0$ for every point $z' \in M_j^{\perp}$, which gives rise to

$$\left\langle z_\infty, z' \right\rangle = \left\langle \lim_{k\to\infty} z_{\Delta_k(i)}, z' \right\rangle = \lim_{k\to\infty} \left\langle z_{\Delta_k(i)}, z' \right\rangle = 0. \tag{A6}$$

Therefore, there exists $z_\infty \in M_j$ for each $j \in \{1, 2, \dots, J\}$, and hence we have $z_\infty \in M = \bigcap_1^J M_j$.

Finally, $z_\infty = P_M(z_0)$ is going to be obtained to finish the proof. To show that z_∞ is the orthogonal projection of z_0 onto M, it suffices to show that $z_0 - z_\infty \in M^\perp$, because it will give rise to

$$z_0 = \overbrace{z_\infty}^{\in M} + \overbrace{z_0 - z_\infty}^{\in M^\perp}, \tag{A7}$$

by the projection theorem in Hilbert spaces. Let $z \in M$, and hence $z \in M_{j_{k+1}}$. Since the projection operator is self-adjoint and idempotent, it can also be proved that $z_k - P_{M_{j_{k+1}}}(z_k) \in M_{j_{k+1}}^\perp$, then

$$\langle z_k - z_{k+1}, z \rangle = \left\langle z_k - P_{M_{j_{k+1}}}(z_k), z \right\rangle = 0, \tag{A8}$$

which yields

$$\langle z_0 - z_\infty, z \rangle = \lim_{k\to\infty} \langle z_0 - z_k, z \rangle = \lim_{k\to\infty} \left(\langle z_0 - z_1, z \rangle + \langle z_1 - z_2, z \rangle + \cdots + \langle z_{k-1} - z_k, z \rangle \right) = 0. \tag{A9}$$

Note that $z \in M$, so we have $z_0 - z_\infty \in M^\perp$, and the proof is complete. □

B PROOF OF PROPOSITION 2

By the definition of the adjoint, it follows that

$$\langle e, Gu \rangle_Q = e^T \mathbf{Q} G \mathbf{R}^{-1} \mathbf{R} u = \left\langle \mathbf{R}^{-1} G^T \mathbf{Q} e, u \right\rangle_R = \langle G^* e, u \rangle_R, \tag{B10}$$

where $\mathbf{R} = diag\{R, R, \dots, R\} \in \mathbb{R}^{\ell \cdot N \times \ell \cdot N}$ and $\mathbf{Q} = diag\{Q, Q \dots, Q\} \in \mathbb{R}^{\mathrm{m \cdot N \times m \cdot N}}$. The update law (47) can now be written as

$$u_{k+1} = u_k + \mathbf{R}^{-1} G^T \mathbf{Q} e_{k+1}, \tag{B11}$$

i.e.

$$u_{k+1}(t) = u_k(t) + \sum_{i=t+1}^{N} R^{-1} B^T \left(A^T\right)^{i-t-1} C^T Q e_{k+1}(i). \tag{B12}$$

Then, set $p_{k+1}(t)$ as

$$p_{k+1}(t) = \sum_{i=t+1}^{N} \left(A^T\right)^{i-t-1} C^T Q e_{k+1}(i), \tag{B13}$$

which yields (55), and hence for $t \in [0,\ N-1]$, $p_{k+1}(t)$ can be computed by the recursion relation

$$p_{k+1}(t) = A^T p_{k+1}(t+1) + C^T Q e_{k+1}(t+1), \tag{B14}$$

with the boundary condition $p_{k+1}(N) = 0$. As in the work of Amann et al,[32] assume that the state of system (1) is fully known, then there exists a causal implementation with respect to $p_{k+1}(t)$ in the form (56). It now follows from (1), (56) and (B14) that

$$\begin{aligned} x_{k+1}(t+1) - x_k(t+1) &= A\left[x_{k+1}(t) - x_k(t)\right] + BR^{-1}B^T p_{k+1}(t) \\ &= \left[I_n + BR^{-1}B^T K(t)\right]^{-1} A\left[x_{k+1}(t) - x_k(t)\right] + BR^{-1}B^T \xi_{k+1}(t). \end{aligned} \tag{B15}$$

Furthermore, to eliminate $p_{k+1}(t)$, substituting (56) and (B15) to (B14) yields

$$f_1\left[X, K(t), K(t+1)\right] \cdot \left[x_{k+1}(t+1) - x_k(t+1)\right] = f_2\left[X, K(t+1), \xi_{k+1}(t), \xi_{k+1}(t+1), e_k(t+1)\right], \tag{B16}$$

where $f_1(\cdot)$ and $f_2(\cdot)$ are functions of their arguments and $X = \left\{A, B, C, Q, R^{-1}\right\}$. If both $f_1(\cdot)$ and $f_2(\cdot)$ are set equal to 0, (B16) holds independently of system state and it gives rise to the Riccati equation (53) and the difference equation (54), respectively. Finally, according to $p_{k+1}(N) = 0$, if both $K(N)$ and $\xi_{k+1}(N)$ are also set equal to 0, (56) still holds independent of system state when $t = N$.

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