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FALSENESS OF THE FINITENESS PROPERTY OF THE SPECTRAL SUBRADIUS

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We prove that there exist infinitely may values of the real parameter α for which the exact value of the spectral subradius of the set of two matrices (one matrix with ones above and on the diagonal and zeros elsewhere, and one matrix with α below and on the diagonal and zeros elsewhere, both matrices having two rows and two columns) cannot be calculated in a finite number of steps. Our proof uses only elementary facts from the theory of formal languages and from linear algebra, but it is not constructive because we do not show any explicit value of α that has described property. The problem of finding such values is still open.

Keywords: finiteness property, spectral subradius

1. Introduction

One of the most important ideas in control theory is the absolute asymptotic stability of linear discrete systems. This idea is strongly related to the value of the spectral radii of sets of matrices that describe possible behaviour of such systems. There are two generalizations of the idea of the spectral radius that can be found in many publications. Those are the joint spectral radius ($\hat{\rho}$) and the generalized spectral radius ($\overline{\rho}$), given respectively by the following formulae:

$$\hat{\rho}(\Sigma) = \overline{\lim}\,[\sup\{\rho(A) : A \in \Sigma^n\}]^{\frac{1}{n}} = \sup_{n\in\mathbb{N}} \sup_{A\in\Sigma^n} \rho(A)^{\frac{1}{n}},$$

$$\overline{\rho}(\Sigma) = \overline{\lim}\,[\sup\{\|A\| : A \in \Sigma^n\}]^{\frac{1}{n}} = \inf_{n\in\mathbb{N}} \sup_{A\in\Sigma^n} \|A\|^{\frac{1}{n}},$$

where Σ is the set of $k \times k$ matrices and

$$\Sigma^n = \{A_n A_{n-1} A_{n-2} \dots A_1 : A_i \in \Sigma\}.$$

Recall that for all bounded sets of matrices we have (Elsner, 1995):

$$\hat{\rho}(\Sigma) = \overline{\rho}(\Sigma)$$

and this common value is denoted by $\rho(\Sigma)$. But even when we know it, it is still hard to compute the value of the spectral radius of a set of matrices because the above formulae include limits and suprema that can be rarely calculated in an analytical way. Therefore, any algorithm that produces an exact or well estimated value of the spectral radius in a finite number of steps would be of paramount importance. One of the concepts that could lead to such an algorithm was the so-called finiteness conjecture (Lagarias and Wang, 1995). It says that for every finite set of matrices Σ there is a natural number k such that

$$\rho(\Sigma) = \sup_{A\in\Sigma^k} \rho(A)^{\frac{1}{k}}.$$

It is obvious that if this conjecture is true, then we can construct an algorithm that will give us an exact value of the spectral radius of any finite set of matrices in a finite number of steps. The only problem that is related to this algorithm is the computation of the number of steps that we have to perform to complete the whole process. As the truth of the above conjecture could form a basis for important numerical methods, it was strongly investigated. Tsitsiklis and Blondel (1997) proved that the computation of the spectral radius could be a very hard task. Unfortunately, the finiteness conjecture appeared to be false (Blondel *et al.*, 2003).

There are several types of the asymptotic stability of discrete linear systems except the above-mentioned absolute asymptotic stability, which is the strongest of them all. It is worth considering the so-called selective asymptotic stability, which can be defined as follows.

Definition 1. Let Σ be the set of matrices and

$$x(t+1) = f(t)x(t),$$

where $x(0) = x_0$ and $f(t) : \mathbb{N} \to \Sigma$ and $x(x_0, f, t)$ is its solution. We say that such a system is *selectively asymptotically stable* when a function f exists such that

$$\lim_{t\to\infty} x(x_0, f, t) = 0.$$

The idea of the selective stability of discrete linear systems is related to the value of the spectral subradius of the set of matrices (Czornik, 2005). As for the spectral radius, there exist generalized and joint spectral radii that are respectively given by

$$\overline{\rho}_*(\Sigma) = \underline{\lim}\,[\inf\{\|A\| : A \in \Sigma^n\}]^{\frac{1}{n}} = \inf_{n\in\mathbb{N}} \inf_{A\in\Sigma^n} \|A\|^{\frac{1}{n}}$$

and

$$\hat{\rho}_*(\Sigma) = \underline{\lim}\,[\inf \rho(A) : A \in \Sigma^n]^{\frac{1}{n}} = \inf_{n\in\mathbb{N}} \inf_{A\in\Sigma^n} \rho(A)^{\frac{1}{n}}.$$

Those values are equal for any nonempty set of matrices and their common value is denoted by $\rho_*(\Sigma)$ (Czornik, 2005), but it is still hard to compute them in an analytical way. Since a discrete linear system related to the set Σ is selectively stable if and only if $\rho_*(\Sigma) < 1$ (Czornik, 2005), for a spectral radius it is worth considering whether there exists a natural number k such that

$$\rho_*(\Sigma) = \inf_{A\in\Sigma^n} \rho(A)^{\frac{1}{n}}.$$

We shall call this hypothesis the finiteness property of the spectral subradius, and we shall prove that it is false, too. The proposed proof will be very similar to that of the falsiness of the finiteness conjecture for the spectral radius (Blondel *et al.*, 2003), but some parts of it will be quite different, so that we show here only the altered parts of the proof and give references to lemmas in other publications.

2. Main Results

Define the set $\Sigma = \{A_0, A_1\}$, where

$$A_0 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

and

$$A_1 = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}.$$

Set

$$\rho_\alpha = \rho_*\big(\{A_0, \alpha A_1\}\big).$$

As $\rho_\alpha > 0$, we can write

$$A_0^\alpha = \frac{1}{\rho_\alpha} A_0, \quad A_1^\alpha = \frac{\alpha}{\rho_\alpha} A_1.$$

Consider a two-letter alphabet $I = \{0, 1\}$ and let

$$I^+ = \{0, 1, 00, 01, 10, 11, 000, \dots\}$$

be the set of finite nonempty words. Let also $I^* = I^+ \cup \{\emptyset\}$. The length of a word $w = w_1 \dots w_t \in I^*$ is equal to $t \geq 0$ and it is denoted by $|w|$. The mirror image of the word w is defined as

$$\overline{w} = w_t \dots w_1 \in I^*.$$

Any word that is identical to its mirror image will be said to be a palindrome. In particular, an empty word is a palindrome, too. For any words $u, v \in I^*$, we write $u > v$ if $u_i = 1$ and $v_i = 0$ for some $i \geq 1$ and $u_j = v_j$ for all $j < i$. This is only a partial order because, e.g., the words 101000 and 1010 cannot be compared. The relation $u < v$ for $u, v \in I^*$ is defined analogously. For any infinite word U, we denote by $F(U)$ the set of all finite factors of the word U. We also say that two words $u, v \in I^+$ are essentially equal if the periodic infinite words $U = uuu\dots$ and $V = vvv\dots$ can be decomposed as $U = xwww\dots$ and $V = ywww\dots$ for some $x, y, w \in I^+$. Words that are not essentially equal are essentially different.

Now a word $w = w_1 w_2 \dots w_t \in I^+$ can be associated with the products $A_w = A_{w_1} A_{w_2} \dots A_{w_t}$ and $A_w^\alpha = A_{w_1}^\alpha A_{w_2}^\alpha \dots A_{w_t}^\alpha$. A word $w \in I^+$ is be said to be optimal for some α if $\rho(A_w^\alpha) = 1$. Define

$$J_w = \{\alpha : \rho(A_w^\alpha) = 1\}.$$

Lemma 1. *Let $u, v \in I^+$ be two words that are essentially different. Write $U = uuu\dots$ and $V = vvv\dots$. Then in the set $F(U) \cup F(V)$ a pair of words $0p0$ and $1p1$ exists such that $p \in I^*$ is a palindrome.*

Proof. See Lemma 3.1 in (Blondel *et al.*, 2003). ∎

Lemma 2. *Let $u, v \in I^+$ be two essentially different words, and let $U = uuu\dots$ and $V = vvv\dots$. Then there exist words $a, b, x, y \in I^+$ satisfying $|x| = |y|$, $x < y$, $\overline{x} < \overline{y}$, $x < \overline{y}$, $\overline{x} < y$ and a palindrome $p \in I^*$ such that $U = apxpxpx\dots$ and $V = bpypypy\dots$, or one of the words U and V (U, say) can be decomposed as*

$$U = apxpxpx\cdots = bpypypy\dots$$

Proof. By Lemma 1 a pair of words $0p0$ and $1p1$ exists in the set $F(U) \cup F(V)$ such that p is a palindrome. Without loss of generality, we can assume that $1p1$ occurs in U.

Then it occurs in U infinitely many times because U is an infinite periodic word. Write

$$U = a'1p1d1p1d\dots$$

and, analogously,

$$W = b'0p0f0p0f\dots,$$

where W is either U or V. With no loss of generality, we may assume that $|d| = |f|$. Otherwise, we can always take $d' = d1p1d1\dots1p1d$ instead of d and $f' = f0p0f\dots0p0f$ instead of f in such a way that $|f'| = |d'|$. Now we only have to set $a = b'0$, $b = a'1$, $x = 0f0$, $y = 1d1$, which completes the proof. ■

Definition 2. We say that a matrix A is *dominated* by a matrix B if $A \le B$ componentwise and $\operatorname{tr}(A) < \operatorname{tr}(B)$.

For all words w, the matrix A_w satisfies $\det A_w = 1$ and $\operatorname{tr} A_w \ge 2$. We therefore have $\rho(A_u) < \rho(A_v)$ whenever A_u is dominated by A_v.

Lemma 3. *For any word $w \in I^+$ we have*

$$A_{\overline{w}} - A_w = k(w)T,$$

where k(w) is an integer and

$$T = A_0A_1 - A_1A_0 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

Moreover, $k(w)$ is positive if and only if $w > \overline{w}$.

Proof. See Lemma 4.2 in (Blondel *et al.*, 2003). ■

Lemma 4. *Let $A, B \in \mathbb{R}^{2\times 2}$, $\det A \neq 0$, $\det B = 0$ and $B \neq 0$. Then $AB \neq 0$.*

Proof. Let

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

Since $\det B = 0$ and $B \neq 0$, we have

$$B = \begin{bmatrix} p & q \\ \beta p & \beta q \end{bmatrix}$$

and $p \neq 0$ or $q \neq 0$. Therefore,

$$AB = 0 \iff \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} p & q \\ \beta p & \beta q \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

Thus

$$\begin{bmatrix} p(a+b\beta) & q(a+b\beta) \\ p(c+d\beta) & q(c+d\beta) \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

We have rejected the case when $p = 0$ and $q = 0$, so assume without loss of generality that $p = 0$ and $q \neq 0$. Then

$$q(a+b\beta) = 0 \iff a + b\beta = 0 \iff a = -b\beta,$$

$$q(c+d\beta) = 0 \iff c + d\beta = 0 \iff c = -d\beta.$$

But it follows that

$$0 \neq \det A = \det\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \det\begin{bmatrix} -b\beta & b \\ -d\beta & d \end{bmatrix} = 0,$$

which is a contradiction. Thus $AB \neq 0$ when $p = 0$ and $q \neq 0$. The proofs that $AB \neq 0$ when ($p \neq 0$ and $q = 0$) or ($p \neq 0$ and $q \neq 0$) are analogous. ■

Lemma 5. *Let $A, B \in \mathbb{R}^{2\times 2}$, $\det B \neq 0$, $\det A = 0$ and $A \neq 0$. Then $AB \neq 0$.*

Proof. The proof is analogous to that of Lemma 4. ■

Lemma 6. *For any word of the form $w = psq$, where $s < \overline{s}$ and $q > \overline{p}$, the matrix A'_w with $w' = p\overline{s}q$ is dominated by A_w.*

Proof. By Lemma 3 we have

$$\begin{aligned} A'_w - A_w &= A_pA_{\overline{s}}A_q - A_pA_sA_q = A_p(A_{\overline{s}} - A_s)A_q \\ &= k(s)A_pTA_q \end{aligned}$$

and $k(s) \le 0$. Assume that $k(s) = 0$. Then

$$k(s)A_pTA_q = 0,$$

but

$$k(s)A_pTA_q = A_p(A_{\overline{s}} - A_s)A_q.$$

The matrices A_p and A_q are finite products of the matrices A_0 and A_1, which are nonsingular. Therefore, the matrices A_p and A_q are nonsingular, too. Since $s \neq \overline{s}$, we have that $A_{\overline{s}} - A_s \neq 0$. When the latter matrix is nonsingular, so is the matrix

$$A_p(A_{\overline{s}} - A_s)A_q.$$

Thus it cannot be the zero matrix and $k(s) \neq 0$. When the matrix $A_{\overline{s}} - A_s$ is singular but nonzero, we can use Lemmas 4 and 5 to write

$$A_p(A_{\overline{s}} - A_s)A_q \neq 0$$

and therefore $k(s) \neq 0$, so that $k(s) < 0$.

Now we can observe that for $i = 0, 1$ we have

$$A_iTA_i = T = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

and

$$A_0TA_1 = \begin{bmatrix} 0 & -1 \\ -1 & -1 \end{bmatrix}.$$

Since A_0TA_1 is dominated by A_iTA_i for $i = 0, 1$, the proof is complete. ■

Definition 3. If $w = psq$ and the matrix $A_{p\overline{s}q}$ dominates the matrix A_{psq}, then the word $\overline{s}$ is called the *dominating flip* of the word s.

Lemma 7. *Let $u, v \in I^+$ be two words that are essentially different. Then $J_u \cap J_v = \emptyset$.*

Proof. Let $u, v \in I^+$ be two words that are essentially different. In order to prove the result, we show that if $\rho(A_u^\alpha) = \rho(A_v^\alpha)$ for some value of α, then there is a word w satisfying $\rho(A_w^\alpha) < \rho(A_u^\alpha)$.

By Lemma 2, there exist words $a, b, x, y \in I^+$ satisfying $|x| = |y|$, $x < y$, $\overline{x} < \overline{y}$, $x < \overline{y}$, $\overline{x} < y$, and a palindrome $p \in I^*$ such that

$$U = apxpxpx\ldots \quad \text{and} \quad V = bpypypy\ldots$$

or

$$U = apxpxpx\cdots = bpypypy\ldots.$$

Since neither U nor V is equal to $000\ldots$ or $111\ldots$, the matrices A_{xp} and A_{yp} are strictly positive.

Consider the word $xpxpxpypypyp$. Setting $s = xpy$, we make a dominating flip in this word and get the word $xpxp\overline{y}p\overline{x}pypyp$. Then we set $s = xp\overline{y}p\overline{x}py$ and make another dominating flip. As a result, by Lemma 6 the matrix $A_{xpxpxpypypyp}$ is dominated by the matrix $A_{xp\overline{y}pxpy\overline{x}pyp}$. Analogously, any matrix $A_sA_vA_r$, $v \in I^*$ is dominated by the matrix $A_{s'}A_vA_{r'}$, where $s = xpxpxp$, $r = ypypyp$, $s' = xp\overline{y}pxp$ and $r' = yp\overline{x}pyp$.

Let us denote by L and L' the linear operators $A \to A_sAA_r$ and $A \to A_{s'}AA_{r'}$ acting in $\mathbb{R}^4$ as well as their 4×4 matrices. It is known that $L = A_r^T \otimes A_s$ and $L' = (A_r')^T \otimes A_s'$, where $\otimes$ denotes the Kronecker product (Horn and Johnson, 1991, Lem. 4.3.1). Both L and L' are strictly positive. The minimal closed convex cone in $\mathbb{R}^4$ containing all matrices A_v, $v \in I^*$, is the cone of all nonnegative 2×2 matrices of the form βA_w, $\beta > 0$, $w \in I^*$. In particular, this is true for the matrices

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

Hence $L' \le L$ elementwise and $L \ne L'$. From the Perron-Frobenius theory (see Problem 8.15 in (Horn and Johnson, 1985)) we get $\rho(L') < \rho(L)$. The spectral radius of a Kronecker product is the product of the spectral radii (Horn and Johnson, 1991, Th. 4.2.12), and hence $\rho(L') = \rho(A_{s'})\rho(A_{r'}) < \rho(A_s)\rho(A_r) = \rho(L)$. Since the flips performed do not change the average proportion of the matrices A_0 and A_1 in the product, we can also write

$$\rho(L^\alpha) = \rho(A_s^\alpha)\rho(A_r^\alpha)$$

and

$$\rho(L^{'\alpha}) = \rho(A_{s'}^\alpha)\rho(A_{r'}^\alpha)$$

for each $\alpha > 0$, where L^α and $L^{'\alpha}$ are defined analogously to L and L', respectively.

Suppose that $\rho(A_u^\alpha) = \rho(A_v^\alpha) = 1$. Then $\rho(A_s^\alpha) = \rho(A_r^\alpha) = 1$, and hence either $\rho(A_{s'}^\alpha) < 1$ or $\rho(A_{r'}^\alpha) < 1$, which is a contradiction. This completes the proof of the lemma. ■

Now we are ready to prove the main result of this paper.

Theorem 1. *There are uncountably many values of the real parameter α for which the pair of matrices*

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \quad \alpha \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$$

does not satisfy the finiteness property of the spectral subradius.

Proof. We have

$$J_w \cap (0,1) = \{\alpha \in (0,1) : \rho(A_w^\alpha) = 1\}$$

or, equivalently,

$$J_w \cap (0,1) = \Big\{\alpha \in (0,1) : \Big(\rho(A_w)\alpha^{|w|_1}\Big)^{\frac{1}{|w|}} = \inf_{v \in I^+} \Big(\rho(A_v)\alpha^{|v|_1}\Big)^{\frac{1}{v}}\Big\}, \quad (1)$$

where $|w|_1$ denotes the number of the letters '1' in the word w. Now we define the function

$$h(w,\beta) = \frac{1}{|w|}\big(\ln\rho(A_w) + |w|_1\beta\big)$$

associated with $w \in I^+$, and let

$$h(\beta) = \inf_{w \in I^+} h(w,\beta).$$

Passing on to the logarithmic scale in the expression (1), we get

$$J_w \cap (0,1) = \{e^\beta : \beta \in \mathbb{R}, h(w,\beta) = h(\beta)\} \cap (0,1). \quad (2)$$

The functions $h(w,\beta)$ are affine and $h(\beta) \le h(w,\beta)$ for all $w \in I^+$ and $\beta \in \mathbb{R}$, so we only have to show that h is convex and continuous. Let us start with the proof of the continuity of the function $h(w,\beta)$.

The function

$$h(w,\beta) = \frac{1}{|w|}\big(\ln\rho(A_w) + |w|_1\beta\big)$$

is not continuous only in the case when $|w| = 0$ or $\rho(A_q) = 0$. We do not have to care about the case when

$|w| = 0$ because $|w| = 0$ only for an empty word that does not belong to the set I^+.

Let us consider the eigenvalues of the 2×2 matrix in the general case:

$$\det\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}\right) = \det\begin{bmatrix} a-\lambda & b \\ c & d-\lambda \end{bmatrix} = (a-\lambda)(d-\lambda) - bc = ad - d\lambda - a\lambda + \lambda^2 - bc = \lambda^2 - (a+d)\lambda + ad - bc = 0.$$

The spectral radius of the matrix considered will be zero only when both of the solutions to the above equation are equal (i.e., $\lambda_1 = \lambda_2$), and therefore

$$(a+d)^2 - 4(ad-bc) = 0.$$

The second important condition is that the trace of the considered matrix has to be zero. Therefore,

$$a + d = 0 \Longrightarrow a = -d$$

and

$$(a+d)^2 - 4(ad-bc) = 0 - 4(ad-bc) = 0.$$

Hence

$$ad - bc = 0,$$

which is equivalent to

$$-a^2 - bc = 0 \Longrightarrow a^2 = -bc.$$

This implies that the numbers b and c have opposite signs or $a = 0$ and ($b = 0$ or $c = 0$). There exists no product of the matrices A_0 and A_1 that includes negative numbers or zeros on the main diagonal. Therefore, the spectral radius of any product of the matrices A_0 and A_1 is not equal to zero, and this proves that the function $h(w,\beta)$ is continuous.

Now we are ready to prove that the function $h(\beta)$ is continuous, too. Assume that

$$\inf_{w\in I^+} h(w,\beta_1) - \inf_{w\in I^+} h(w,\beta_2) > 0.$$

Then for $|\beta_1 - \beta_2| < \delta$ we have

$$\begin{aligned}
&\left|\inf_{w\in I^+} h(w,\beta_1) - \inf_{w\in I^+} h(w,\beta_2)\right| \\
&\le \left|\sup_{w\in I^+} h(w,\beta_1) - \inf_{w\in I^+} h(w,\beta_2)\right| \\
&= \left|\sup_{w\in I^+} (h(w,\beta_1) - h(w,\beta_2))\right| \\
&= \left|\sup_{w\in I^+}\left(\frac{1}{|w|}\big(\ln\rho(A_w) + |w|_1\beta_1\big) - \frac{1}{|w|}\big(\ln\rho(A_w) + |w|_1\beta_2\big)\right)\right| \\
&= \left|\sup_{w\in I^+}\left(\frac{|w|_1}{|w|}(\beta_1 - \beta_2)\right)\right| \le |\beta_1 - \beta_2| < \delta.
\end{aligned}$$

Assume now that

$$\inf_{w\in I^+} h(w,\beta_1) - \inf_{w\in I^+} h(w,\beta_2) < 0.$$

Then for $|\beta_1 - \beta_2| < \delta$ we have

$$\begin{aligned}
&\left|\inf_{w\in I^+} h(w,\beta_1) - \inf_{w\in I^+} h(w,\beta_2)\right| \\
&\le \left|\sup_{w\in I^+} h(w,\beta_2) - \inf_{w\in I^+} h(w,\beta_1)\right| \\
&= \left|\sup_{w\in I^+} (h(w,\beta_2) - h(w,\beta_1))\right| \\
&= \left|\sup_{w\in I^+}\left(\frac{1}{|w|}\big(\ln\rho(A_w) + |w|_1\beta_2\big) - \frac{1}{|w|}\big(\ln\rho(A_w) + |w|_1\beta_1\big)\right)\right| \\
&= \left|\sup_{w\in I^+}\left(\frac{|w|_1}{|w|}(\beta_2 - \beta_1)\right)\right| \\
&\le |\beta_2 - \beta_1| = |\beta_1 - \beta_2| < \delta.
\end{aligned}$$

Therefore, for all $\epsilon > 0$ there exists $\delta = \epsilon$ such that

$$|h(\beta_1) - h(\beta_2)| < \epsilon$$

when $|\beta_1 - \beta_2| < \delta$, and thus the function $h(\beta)$ is continuous.

The last thing that we have to prove before we proceed to the next part of the proof is that the function $h(\beta)$ is convex. Let $t \in [0,1]$. Then

$$\begin{aligned}
&(1-t)h(\beta_1) + th(\beta_2) \\
&= (1-t)\inf_{w\in I^+} h(w,\beta_1) + t\inf_{w\in I^+} h(w,\beta_2) \\
&= (1-t)\inf_{w\in I^+}\frac{1}{|w|}\big(\ln\rho(A_w) + |w|_1\beta_1\big) \\
&\quad + t\inf_{w\in I^+}\frac{1}{|w|}\big(\ln\rho(A_w) + |w|_1\beta_2\big) \\
&= \inf_{w\in I^+}\frac{\ln\rho(A_w)}{|w|} + \inf_{w\in I^+}\frac{|w|_1\beta_1}{|w|} - t\inf_{w\in I^+}\frac{\ln\rho(A_w)}{|w|} \\
&\quad - t\inf_{w\in I^+}\frac{|w|_1\beta_1}{|w|} + t\inf_{w\in I^+}\frac{\ln\rho(A_w)}{|w|} \\
&\quad + t\inf_{w\in I^+}\frac{|w|_1\beta_2}{|w|}
\end{aligned}$$

$$
\begin{aligned}
&= \inf_{w\in I^+} \frac{\ln\rho(A_w)}{|w|} + \beta_1 \inf_{w\in I^+} \frac{|w|_1}{|w|} - t\beta_1 \inf_{w\in I^+} \frac{|w|_1}{|w|} \\
&\quad + t\beta_2 \inf_{w\in I^+} \frac{|w|_1}{|w|} \\
&= \inf_{w\in I^+} \frac{\ln\rho(A_w)}{|w|} + \inf_{w\in I^+} \frac{|w|_1}{|w|}(\beta_1 - t\beta_1 + t\beta_2) \\
&= \inf_{w\in I^+} \frac{\ln\rho(A_w)}{|w|} + \inf_{w\in I^+} \frac{|w|_1}{|w|}((1-t)\beta_1 + t\beta_2) \\
&= \inf_{w\in I^+} \left(\frac{\ln\rho(A_w)}{|w|} + \frac{|w|_1}{|w|}((1-t)\beta_1 + t\beta_2) \right) \\
&= \inf_{w\in I^+} \frac{1}{|w|} \left(\ln\rho(A_w) + ((1-t)\beta_1 + t\beta_2)|w|_1\right) \\
&= \inf_{w\in I^+} h(w, (1-t)\beta_1 + t\beta_2) = h((1-t)\beta_1 + t\beta_2),
\end{aligned}
$$

and this implies that the function $h(\beta)$ is convex.

The above deliberations prove that the set

$$\{\beta \in \mathbb{R} : h(w,\beta) = h(\beta)\}$$

is an interval of the real line. This interval is the zero set of a continuous function, and it is therefore closed.

From (2) we conclude that the set $J_w \cap [0,1]$ is a closed subinterval of $[0,1]$. Let us finally show that $(0,1)$ cannot be covered by countably many disjoint closed intervals H_i for $i \geq 1$ (possibly single points). One can find in (Blondel *et al.*, 2003) the proof that the interval $[0,1]$ cannot be covered by countably many disjoint intervals H_i for $i \geq 1$ (possibly single points) unless it is a single interval. Thus the interval $(0,1)$ cannot be covered by countably many disjoint intervals, either. Observe that

$$[0,1] = (0,1) \cup \{0,1\},$$

and if we were able to cover the interval $(0,1)$ by countably many disjoint intervals, then we could cover $[0,1]$ with the same intervals and the points $\{0\}$ and $\{1\}$, which makes a contradiction and completes the proof of the whole theorem. ■

3. Conclusion

Since, in general, the finiteness property of the spectral subradius is false, there are still many particular cases when it is true. For example, the spectral radius of any finite set of diagonal matrices can be calculated in a finite number of steps. It is worth remarking that the falseness of the finiteness property of the spectral subradius does not imply that no algorithm exists that permits to compute an exact value of the spectral subradius of a finite set of real matrices in a finite number of steps. It only shows that it is impossible to set forth such an algorithm in the way that is suggested in the finiteness property of the spectral subradius, and therefore the problem of inventing such an algorithm is still open.

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