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EFFECT OF ANGULAR SPEED VARIATIONS ON THE NONLINEAR VIBRATIONS OF A ROTATIONAL SPRING-MASS SYSTEM

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A rotating spring-mass system is considered using polar coordinates. The system contains a cubic nonlinear spring with damping. The angular velocity harmonically fluctuates about a mean velocity. The dimensionless equations of motion are derived first. The velocity fluctuations resulted in a direct and parametric forcing terms. Approximate analytical solutions are sought using the Method of Multiple Scales, a perturbation technique. The primary resonance and the principal parametric resonance cases are investigated. The amplitude and frequency modulation equations are derived for each case. By considering the steady state solutions, the frequency response relations are derived. The bifurcation points are discussed for the problems. It is found that speed fluctuations may have substantial effects on the dynamics of the problem and the fluctuation frequency and amplitude can be adjusted as passive control parameters to maintain the desired responses.

Key words: spring-mass system, nonlinear rotational vibrations**,** velocity fluctuations, resonances, method of multiple scales.

1. Introduction

Velocity fluctuations affect substantially motion of the dynamical systems. Fluctuation amplitudes and frequencies may be adjusted to obtain the desired responses for the specific systems. The effect of velocity fluctuations on the vibrations of axially moving systems have been extensively studied. In a pioneering study of Pakdemirli *et al.* [1] a numerical study was conducted to determine the stability charts. Principal parametric resonances were investigated for axially moving beams by Öz *et al.* [2]. Pakdemirli and Öz [3] conducted an infinite mode analysis and showed that the velocity fluctuations may lead to extremely complex dynamical phenomena involving resonances up to four modes of vibration. Both tension and speed oscillations were considered for axially moving materials [4]. Harmonic variations in velocity were treated for axially moving laminated composite beams by Ghayesh *et al.* [5]. Experimental investigation of the moving fabric problem for axially accelerating transport velocity was conducted by Lin *et al.* [6]. Chaotic motion was also observed for axially moving systems with fluctuating velocities [7, 8]. Internal resonances and resonances of sum type were studied for viscoelastic beams with pulsating speed [9, 10]. Stabilizing the chatter vibrations in machine tools via harmonic variations in the angular speeds was treated by Pakdemirli & Ulsoy [11]. Apart from the velocity fluctuations which alter the dynamics of the systems, surface effects were also proven to be effective on altering the natural frequencies and dynamics of the vibrating systems leading to quasi-periodic and chaotic behavior [12].

In an early study, Linnett [13] studied the rotating spring mass systems under harmonic excitation force. It is shown that the rotation speed highly influences the natural frequencies of the spring-mass systems [14]. The angular speeds were taken to be constant in both of the studies.

In this study, the harmonic angular velocity fluctuations are incorporated to a rotating spring-mass system for the first time. The dimensionless equation of motion is derived first. It is observed that the variations in velocity resulted in forced and parametric excitation terms. The primary resonances and the principle parametric resonances are investigated in detail using the Method of Multiple Scales, a perturbation technique [15]. The problem involves amplitude and phase variations in time requiring an advanced perturbation technique such as multiple scales, averaging method or the newly proposed shift-perturbation method [16]. The mentioned advanced techniques are more successful in predicting the dynamical behavior of such systems since they allow for shifts in the horizontal and vertical directions [16]. The amplitude and phase variation equations in time are derived for both cases and the steady state solutions are depicted via the frequency response curves. The bifurcation points for solutions are discussed. The response of such systems can be passively controlled by adjusting the amplitude and frequency of the fluctuations.

A major problem in rotating machinery is the small eccentricities which lead to undesired vibrations that harm the mechanical system in the long run. Rotating shafts, blades, propellers, automobile wheels, satellite booms, rolling bears, spur gears, machining tools are severely affected by the rotation if small unbalances exist in the system. In contrast, occasionally, one needs to produce vibrations and the most basic system is to rotate an unbalanced mass about a fixed point to enhance vibrations. Some examples may be the vibrational property induced in cellular phones, fruit vibration harvesters, massage equipment etc. In either case, whether the vibrations are suppressed or enhanced, the simplified model is a rotating spring-mass system with a damping which deserves investigation to understand the basic dynamics of such systems. This study incorporates a variable fluctuation angular velocity to passively control the vibrational behavior of such systems.

2. Equation of motion and the perturbation analysis

The sketch of the problem is given in Fig.1.

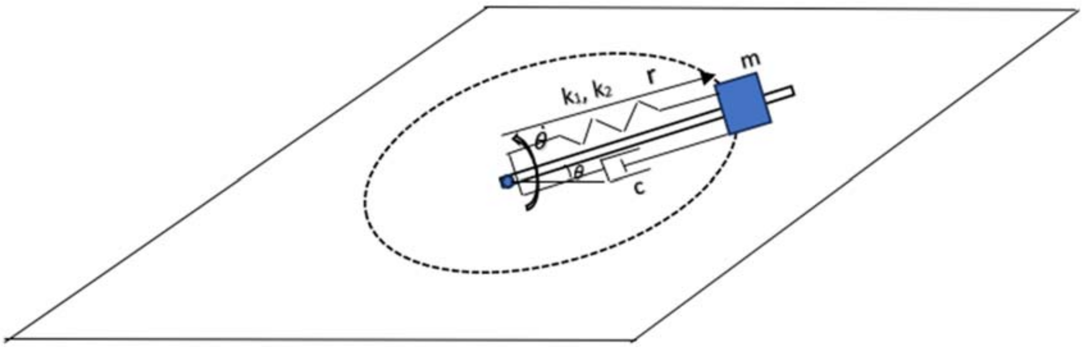

Fig.1. Rotating spring mass system.

The mass is sliding on a frictionless rigid road and rotating in a horizontal plane with fluctuating angular velocity $\dot{\theta}$. The cubic nonlinear spring has a linear spring constant k_1 and a nonlinear spring constant k_2. The viscous damping constant is c. Polar coordinates are employed. In the radial direction, the equation of motion is

$$\ddot{r}^* - r^*\dot{\theta}^2 + \frac{k_1}{m}\left(r^* - r_0\right) + \frac{k_2}{m}\left(r^* - r_0\right)^3 + \frac{c}{m}\dot{r}^* = 0, \tag{2.1}$$

with r^* being the dimensional radial distance from the origin and r_0 being the length of the undeformed spring. The angular velocity is harmonically varying about a mean velocity ω_0

$$\dot{\theta}^{*2} = \omega_0^2\left(1 + \varepsilon f \sin\Omega^* t^*\right), \tag{2.2}$$

where t^* is the dimensional time, εf is the amplitude and Ω^* is the dimensional frequency of the fluctuations. ε is a small book-keeping parameter to ensure that the fluctuations are small compared to the mean velocity. The relative displacement is defined as

$$u^* = r^* - r_0 . \tag{2.3}$$

Substituting (2.2) and (2.3) together with the dimensionless quantities

$$u = \frac{u^*}{r_0}, \qquad t = \omega_0 t^* , \tag{2.4}$$

into (2.1) yields the dimensionless equation of motion

$$\ddot{u} + \omega^2 u + \varepsilon\mu\dot{u} + \varepsilon\alpha_2 u^3 = 1 + \varepsilon f \sin\Omega t + \varepsilon f u \sin\Omega t , \tag{2.5}$$

where

$$\alpha_1 = \frac{k_1}{m\omega_0^2}, \qquad \varepsilon\alpha_2 = \frac{k_2 r_0^2}{m\omega_0^2}, \qquad \varepsilon\mu = \frac{c}{m\omega_0}, \qquad \Omega = \frac{\Omega^*}{\omega_0}, \qquad \omega^2 = \alpha_1 - 1 . \tag{2.6}$$

The damping and nonlinear terms are reordered so that their effects balance the effects of the excitation terms. As can be seen from (2.5), the fluctuations in the angular velocity cause an external excitation term and a parametric excitation term. The initial conditions for the problem are

$$u(0) = 0 , \qquad \dot{u}(0) = 0 . \tag{2.7}$$

An approximate solution will be given for the problem using the method of Multiple Scales, a perturbation technique [15]. Assuming a two-term approximate expansion

$$u(t;\varepsilon) = u_0(T_0, T_1) + \varepsilon u_1(T_0, T_1) + O\left(\varepsilon^2\right), \tag{2.8}$$

with the fast and slow time scales being

$$T_0 = t, \quad T_1 = \varepsilon t , \tag{2.9}$$

and the time derivatives are defined for the new variables

$$\frac{d}{dt} = D_0 + \varepsilon D_1 + \ldots, \qquad \frac{d^2}{dt^2} = D_0^2 + 2\varepsilon D_0 D_1 + \ldots, \tag{2.10}$$

where $D_i = \partial / \partial T_i, \quad i = 0, 1$. Substituting (2.8)-(2.10) into (2.5) and (2.7) and separating terms with respect to their orders yield

$$O(1): D_0^2 u_0 + \omega^2 u_0 = 1 , \tag{2.11}$$

$$u_0(0)=0, D_0u_0(0)=0\,. \tag{2.12}$$

$$O(\varepsilon): D_0^2u_1+\omega^2u_1=-2D_0D_1u_0-\mu D_0u_0-\alpha_2u_0^3+\ f\sin\Omega T_0+fu_0\sin\Omega T_0\,, \tag{2.13}$$

$$u_1(0)=0,\ (D_0u_1+D_1u_0)(0)=0\,. \tag{2.14}$$

The first order solution in terms of complex and real amplitudes is

$$u_0=\frac{1}{\omega^2}+A(T_1)e^{i\omega T_0}+cc=\frac{1}{\omega^2}+a(T_1)\cos\left(\omega T_0+\beta(T_1)\right), \tag{2.15}$$

where *cc* stands for the complex conjugates of the preceding terms and the complex amplitudes are defined as

$$A(T_1)=\frac{1}{2}a(T_1)e^{i\beta(T_1)}\,. \tag{2.16}$$

The initial conditions (2.12) require

$$a(0)=-\frac{1}{\omega^2}\,,\quad \beta(0)=0\,. \tag{2.17}$$

In order to proceed to the next level of approximation, some assumptions are needed for the fluctuation frequencies. The primary resonance case ($\Omega\approx\omega$) and the principal parametric resonance case ($\Omega\approx 2\omega$) will be treated separately.

3. Primary resonances

Primary resonances occur when the velocity fluctuation frequency is near the natural frequency of the system

$$\Omega=\omega+\varepsilon\sigma\,, \tag{3.1}$$

where σ is the detuning parameter expressing the nearness of the external excitation to the natural frequency. Expressing the sin terms in complex forms

$$\sin\Omega t=\frac{e^{i\Omega t}-e^{-i\Omega t}}{2i}\,, \tag{3.2}$$

and substituting (2.15), (3.1) and (3.2) into (2.13) yields

$$D_0^2u_1+\omega^2u_1=-e^{i\omega T_0}\left(2i\omega D_1A+\mu i\omega A+3\alpha_2A^2\bar{A}+\frac{3\alpha_2}{\omega^4}A+\frac{if}{2}\left(1+\frac{1}{\omega^2}\right)e^{i\sigma T_1}\right)+ \\ +NST+cc. \tag{3.3}$$

where *NST* stands for the non-secular terms. Elimination of the secularities yield the complex equation

$$2i\omega D_1 A+\mu i\omega A+3\alpha_2 A^2\bar{A}+\frac{3\alpha_2}{\omega^4}A+\frac{if}{2}\left(1+\frac{1}{\omega^2}\right)e^{i\sigma T_1}=0\,. \tag{3.4}$$

Substituting for the complex amplitudes in (2.16), separating real and imaginary parts and defining a new phase

$$\gamma=\sigma T_1-\beta\,, \tag{3.5}$$

one finally obtains the amplitude and phase modulation equations

$$D_1 a=-\frac{1}{2}\mu a-\frac{f}{2\omega}\left(1+\frac{1}{\omega^2}\right)\cos\gamma\,, \tag{3.6}$$

$$D_1\gamma=\sigma-\frac{3\alpha_2}{2\omega^5}-\frac{3\alpha_2}{8\omega}a^2+\frac{f}{2a\omega}\left(1+\frac{1}{\omega^2}\right)\sin\gamma\,, \tag{3.7}$$

with the initial conditions being

$$a(0)=-\frac{1}{\omega^2}\,,\quad \gamma(0)=0\,. \tag{3.8}$$

For the steady state solutions, $D_1 a=0$ and $D_1\gamma=0$, upon eliminating γ between the equations and solving for the detuning parameter, we get

$$\sigma=\frac{3\alpha_2}{2\omega^5}+\frac{3\alpha_2}{8\omega}a^2\mp\frac{1}{2}\sqrt{\frac{f^2}{\omega^2 a^2}\left(1+\frac{1}{\omega^2}\right)^2-\mu^2}\,, \tag{3.9}$$

or the fluctuation frequency from (3.1) is

$$\Omega=\omega+\varepsilon\sigma=\omega+\varepsilon\left(\frac{3\alpha_2}{2\omega^5}+\frac{3\alpha_2}{8\omega}a^2\mp\frac{1}{2}\sqrt{\frac{f^2}{\omega^2 a^2}\left(1+\frac{1}{\omega^2}\right)^2-\mu^2}\right), \tag{3.10}$$

which is the frequency response equation for the problem. Returning back to the original real variables with the appropriate definitions, the approximate solution in terms of the fluctuation frequency is

$$u=\frac{1}{\omega^2}+a(t)\cos\left(\Omega t-\gamma(t)\right)+O(\varepsilon)\,, \tag{3.11}$$

where the amplitude and phase variations are governed by (3.6)-(3.8). The total radial distance from the origin is

$$r=1+u=1+\frac{1}{\omega^2}+a(t)\cos\left(\Omega t-\gamma(t)\right)+O(\varepsilon)\,. \tag{3.12}$$

A characteristic plot of the frequency response curve (Eq.3.10) for steady state solutions is given in Fig.2. As the fluctuation frequencies are increased, the responses increase up to the point A which is a saddle node bifurcation point. If one increases the frequencies further, then an abrupt decrease in the response is observed with the response decreasing to point C. As fluctuation frequencies are decreased from a higher value starting from point C, an increase in the responses is observed up to point B which is another saddle node bifurcation point. A jump is observed again to the higher curve which is much less compared to the jump between A and C. The response follows the arrows if further decrease is made.

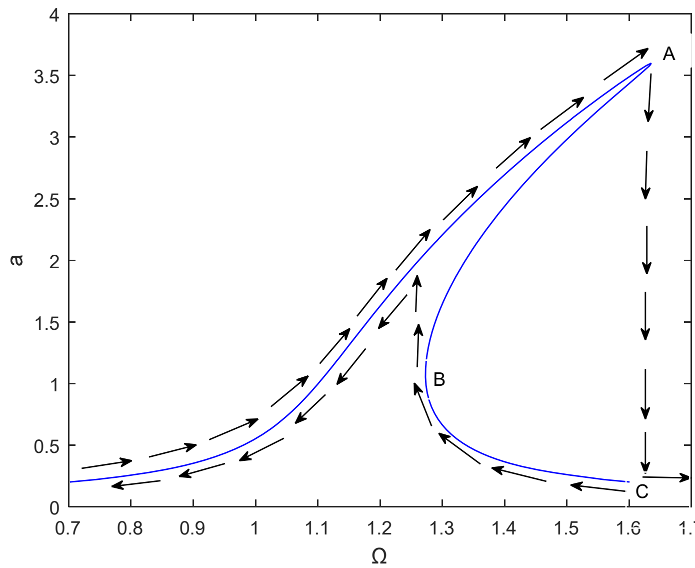

Fig.2. Characteristics of the frequency response curves for primary resonances $\alpha_2 = 1$, $\omega = 1$, $\varepsilon = 0.1$, $f = 0.9$, $\mu = 0.5$.

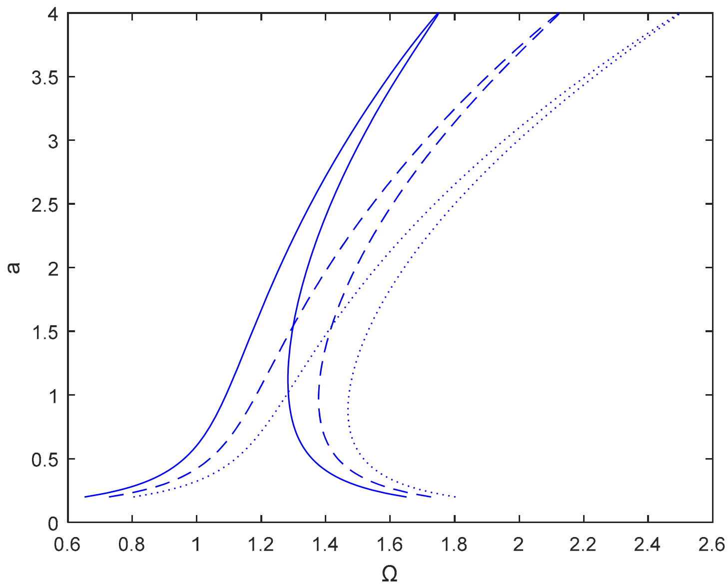

Fig.3. Frequency response curves for primary resonances for $\alpha_2 = 1$, solid; $\alpha_2 = 1.5$, dashed; $\alpha_2 = 2$, dotted $\omega = 1$, $\varepsilon = 0.1$, $f = 1$, $\mu = 0.5$.

The portion of the curve AB is unreachable from both directions and that part of the solution is unstable. This jump phenomenon is a characteristic of nonlinear systems. If the curves are bent to the right, the system has a hardening behavior, else it is softening. In our specific problem, the nonlinearity is of hardening type.

The frequency-response curves are given in Fig.3 for various nonlinearity coefficients. As the nonlinearities increase, the curves bent more to the right without a change in the maximum amplitudes, thereby increasing the region responsible from jump phenomena observed in nonlinear systems.

Effect of fluctuation amplitudes on the responses are depicted in Fig.4. As the fluctuation amplitudes increase, the maximum responses are higher in contrast with the increase in nonlinear coefficients for which the maximum amplitudes remain the same. In both cases, however, an increase results in a wider region of jump phenomenon.

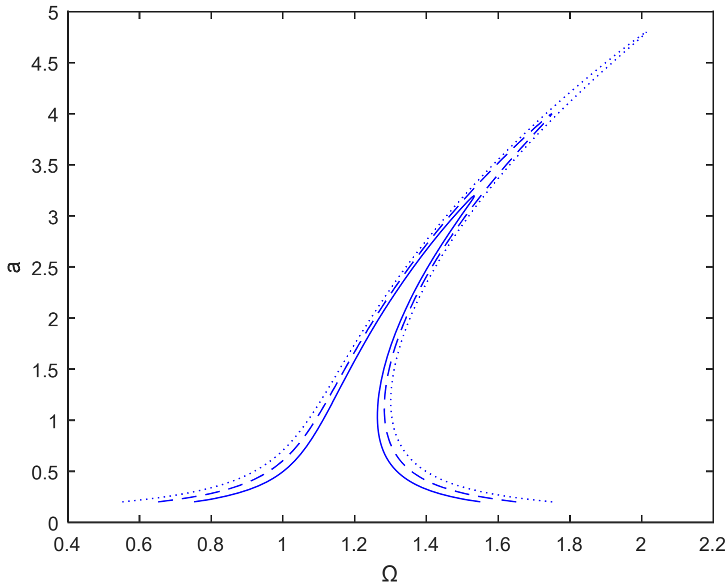

Fig.4. Frequency response curves for primary resonances for $f = 0.8$, solid; $f = 1$, dashed; $f = 1.2$, dotted $\omega = 1,\ \varepsilon = 0.1,\ \alpha_2 = 1,\ \mu = 0.5$.

For similar problems, the steady state solutions of the frequency response curves were contrasted with direct numerical simulations of the system and it was shown that the curves formed by the Method of Multiple Scales produced reliable solutions for small perturbation parameters [17, 18].

4. Principal parametric resonances

Principal parametric resonances occur mainly due to parametric excitation when the velocity fluctuation frequency is near two times the natural frequency of the system

$$\Omega = 2\omega + \varepsilon\sigma\,, \tag{4.1}$$

where σ is the detuning parameter. For this case, Eq.(2.13) reduces to

$$D_0^2 u_1 + \omega^2 u_1 = -e^{i\omega T_0}\left(2i\omega D_1 A + \mu i\omega A + 3\alpha_2 A^2 \bar{A} + \frac{3\alpha_2}{\omega^4} A + \frac{if}{2}\bar{A} e^{i\sigma T_1}\right) + NST + cc\,, \tag{4.2}$$

where *NST* stands for the non-secular terms. Elimination of secularities yield the complex equation

$$2i\omega D_1 A + \mu i\omega A + 3\alpha_2 A^2 \bar{A} + \frac{3\alpha_2}{\omega^4} A + \frac{if}{2} \bar{A} e^{i\sigma T_1} = 0 . \tag{4.3}$$

Substituting for the complex amplitudes in (2.16), separating real and imaginary parts and defining a new phase

$$\gamma = \sigma T_1 - 2\beta , \tag{4.4}$$

one finally obtains the amplitude and phase modulation equations

$$D_1 a = -\frac{1}{2}\mu a - \frac{f}{2\omega} a\cos\gamma , \tag{4.5}$$

$$D_1\gamma = \sigma - \frac{3\alpha_2}{\omega^5} - \frac{3\alpha_2}{4\omega} a^2 + \frac{f}{\omega}\sin\gamma , \tag{4.6}$$

with the same initial conditions given in (3.8).

For the steady state solutions, $D_1 a = 0$ and $D_1\gamma = 0$, and there exists both trivial solutions ($a = 0$) and non-trivial solutions ($a \neq 0$). To find the nontrivial solutions, eliminate γ between the equations and solve for the detuning parameter which is

$$\sigma = \frac{3\alpha_2}{\omega^5} + \frac{3\alpha_2}{4\omega} a^2 \mp \sqrt{\frac{f^2}{\omega^2} - \mu^2} . \tag{4.7}$$

The fluctuation frequency from (4.1) is

$$\Omega = 2\omega + \varepsilon\sigma = 2\omega + \varepsilon\left(\frac{3\alpha_2}{\omega^5} + \frac{3\alpha_2}{4\omega} a^2 \mp \sqrt{\frac{f^2}{\omega^2} - \mu^2} \right) , \tag{4.8}$$

which is the frequency response equation for the non-trivial solutions. Returning back to the original real variables with the appropriate definitions, the approximate solution in terms of the fluctuation frequency is

$$u = \frac{1}{\omega^2} + a(t) cos\left(\frac{\Omega}{2} t - \frac{\gamma(t)}{2} \right) + O(\varepsilon) , \tag{4.9}$$

where the amplitude and phase variations are governed by (4.5) and (4.6). The total radial distance from the origin is

$$r = 1 + u = 1 + \frac{1}{\omega^2} + a(t) cos\left(\frac{\Omega}{2} t - \frac{\gamma(t)}{2} \right) + O(\varepsilon) . \tag{4.10}$$

Results of the principal parametric resonances are qualitatively different from those of primary resonances. A characteristic plot of the frequency response relation (4.8) is given in Fig.5. As the frequency is increased up to point A, only a trivial solution exists which is stable. At point A, a pitchfork bifurcation occurs and a non-trivial and trivial solution co-exists, the former being stable and the latter unstable. Point B is another pitchfork bifurcation point, advancing from where the nontrivial solution is unstable, whereas the trivial solution is stable. For a detailed stability analysis of similar systems, see Öz *et al.* [2].

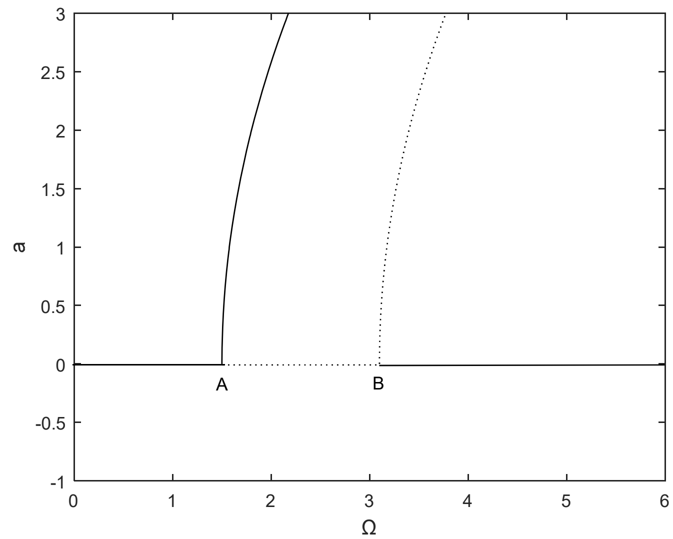

Fig.5. Frequency response curves for principle parametric resonances for stable (solid) and unstable (dotted) solutions $f = 4$, $\omega = 1$, $\varepsilon = 0.2$, $\alpha_2 = 0.5$, $\mu = 0.2$.

Figure 6 depicts the effect of nonlinearities on the frequency-response curves. As the nonlinearities increase, the curves and the bifurcation points shift to the right with the curves bended more.

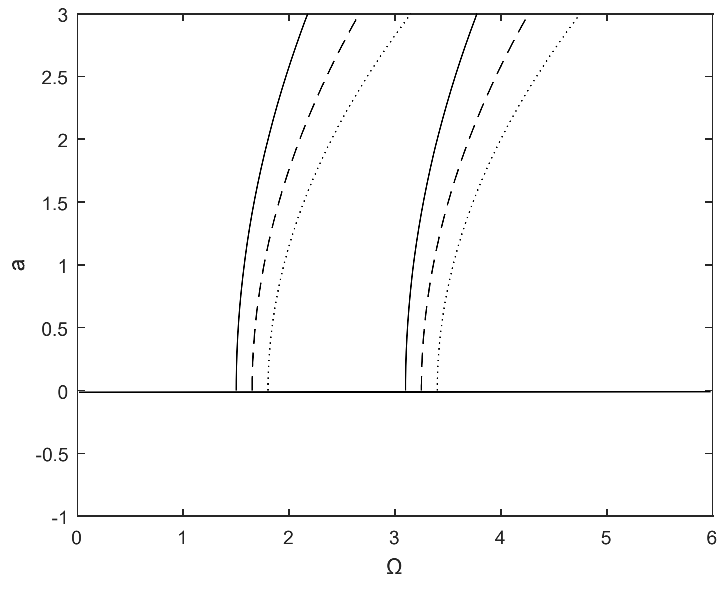

Fig.6. Frequency response curves for principle parametric resonances for $\alpha_2 = 0.5$ (solid), $\alpha_2 = 0.75$ (dashed) and $\alpha_2 = 1$ (dotted) solutions $f = 4$, $\omega = 1$, $\varepsilon = 0.2$, $\mu = 0.2$.

In Fig.7, the effects of fluctuation amplitudes are shown. As the fluctuation amplitudes increase, the bifurcation points diverge from each other with the nontrivial solutions diverging from each other without a change in their bending.

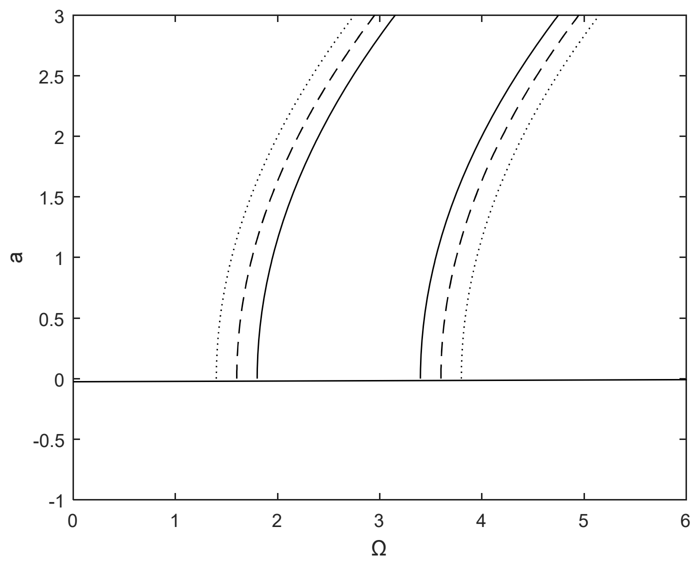

Fig.7. Frequency response curves for principle parametric resonances for $f = 4$ (solid), $f = 5$ (dashed) and $f = 6$ (dotted) solutions $\alpha_2 = 1$, $\omega = 1$, $\varepsilon = 0.2$, $\mu = 0.2$.

5. Concluding remarks

A mathematical model is developed to incorporate angular velocity fluctuations for a rotating spring-mass system. Many physical systems including continuous models (if treated as rigid bodies) can be modelled as a spring-mass idealization. The equations are cast into a dimensionless form for which approximate analytical solutions are presented by using the Method of Multiple Scales, a perturbation technique. The model inherits both external excitation and parametric excitation and the primary and principal parametric resonances are investigated in detail which are the most fundamental resonances for such excitations. From the amplitude phase modulation equations, the steady state solutions are derived in the form of frequency response functions. Effects of amplitudes and frequencies of the velocity fluctuations and the nonlinearities on the responses are depicted in figures in detail.

The nonlinearity is a system parameter which cannot be easily controlled. However, the speed amplitude and frequency of fluctuations can easily be altered in a system. The most important parameter is the frequency of speed fluctuations. Depending on the goal, the system has to be operated in a suitable frequency range. If the goal is to suppress vibrations, the resonance regions should be avoided and they are close to the natural frequency for primary resonances and two times the natural frequency for principal parametric resonances. If the goal is to enhance vibrations, then the resonance regions may be selected with a caution not to harm the whole system by excessive vibrations. The amplitudes of fluctuations are of secondary importance and affect the response curves and their ranges in the resonant regions. Another factor which has a direct influence on the resonance curves is the damping which reduces the responses in the resonant regions. This study underlines the principles of passive control parameters for the rotating-mass systems. An active control of the system can be done in further studies.

Nomenclature

$a(T_1)$ – real amplitudes

$A(T_1)$ – complex amplitudes

c – dimensional viscous damping constant

D_0 – derivative with respect to fast time scale

D_1 – derivative with respect to slow time scale
k_1 – linear spring constant
k_2 – non-linear spring constant
m – mass of the rotating object
r^* – dimensional radial distance
r – non-dimensional radial distance
r_0 – length of the undeformed spring
t^* – dimensional time
t – non-dimensional time
T_0 – fast time scale
T_1 – slow time scale
u^* – dimensional relative displacement
u – non-dimensional relative displacement
u_0 – unperturbed solution
u_1 – correction to perturbed solution
α_1 – non-dimensional linear spring constant
α_2 – non-dimensional non-linear spring constant
$\beta(T_1)$ – phases
γ – defined new phase
ε – a small book-keeping parameter
εf – the amplitude of the fluctuations
$\varepsilon\mu$ – non-dimensional viscous damping constant
θ^* – dimensional angular displacement
θ – non-dimensional angular displacement
$\dot{\theta}^*$ – dimensional angular velocity
$\dot{\theta}$ – non-dimensional angular velocity
σ – detuning parameter
ω_0 – angular mean velocity
ω – non-dimensional natural frequency
Ω^* – dimensional frequency of the velocity fluctuations
Ω – non-dimensional frequency of the velocity fluctuations

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