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APPROXIMATION OF FRACTIONAL POSITIVE STABLE CONTINUOUS–TIME LINEAR SYSTEMS BY FRACTIONAL POSITIVE STABLE DISCRETE–TIME SYSTEMS

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Fractional positive asymptotically stable continuous-time linear systems are approximated by fractional positive asymptotically stable discrete-time systems using a linear Padé-type approximation. It is shown that the approximation preserves the positivity and asymptotic stability of the systems. An optional system approximation is also discussed.

Keywords: Padé approximation, fractional system, linear positive system.

1. Introduction

In positive systems inputs, state variables and outputs take only non-negative values. Examples of positive systems are industrial processes involving chemical reactors, heat exchangers and distillation columns, storage systems, compartmental systems, or water and atmospheric pollution models. A variety of models having positive linear behavior can be found in engineering, management science, economics, social sciences, biology and medicine, etc. Positive linear systems are defined on cones and not on linear spaces. Therefore, the theory of positive systems is more complicated and less advanced. An overview of state of the art in positive systems theory is given by Farina and Rinaldi (2000) as well as Kaczorek (2002).

The stability of positive linear systems was investigated by Farina and Rinaldi (2000) as well as Kaczorek (2002), and that of fractional linear systems by Busłowicz (2008; 2012), Busłowicz and Kaczorek (2009), as well as Kaczorek (2011). The problem of preservation of positivity by approximating the continuous-time linear systems by the corresponding discrete-time linear systems was addressed by Kaczorek (1999).

A linear Padé-type approximation of the exponential matrix of positive asymptotically stable continuous-time linear systems was applied by Kaczorek (2013). It was shown that the approximation preserves the positivity and asymptotic stability of the systems.

In this paper the linear Padé-type approximation will be applied to fractional positive continuous-time linear systems.

The paper is organized as follows. In Section 2, basic definitions and theorems concerning positive standard and fractional continuous-time and discrete-time linear systems are recalled. In Section 3, the main result of the paper is presented. It is shown that the approximation preserves the positivity and asymptotic stability of the systems. Concluding remarks are given in Section 4.

The following notation will be used:

- $\mathbb{R}$: the set of real numbers,
- $\mathbb{R}^{n\times m}$: the set of $n \times m$ real matrices,
- $\mathbb{R}_+^{n\times m}$: the set of $n \times m$ matrices with nonnegative entries and $\mathbb{R}_+^n = \mathbb{R}_+^{n\times 1}$,
- M_n: the set of $n \times n$ Metzler matrices (real matrices with nonnegative off-diagonal entries),
- M_{ns}: the set of $n \times n$ asymptotically stable Metzler matrices,
- I_n: the $n \times n$ identity matrix.

2. Preliminaries and problem formulation

2.1. Standard positive systems. Consider the continuous-time linear system

$$\dot{x}(t) = A_c x(t) + B_c u(t), \quad x(0) = x_0, \tag{1}$$

where $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^m$ are the state and input vectors and $A_c \in \mathbb{R}^{n\times n}$, $B_c \in \mathbb{R}^{n\times m}$.

Definition 1. (*Farina and Rinaldi, 2000; Kaczorek, 2002*) The continuous-time system (1) is called (internally) *positive* if $x(t) \in \mathbb{R}^n_+$, $t \geq 0$ for any initial conditions $x(0) = x_0 \in \mathbb{R}^n_+$ and all inputs $u(t) \in \mathbb{R}^m_+$, $t \geq 0$.

Theorem 1. (Farina and Rinaldi, 2000; Kaczorek, 2002) *The continuous-time system (1) is positive if and only if*

$$A_c \in M_n, \quad B_c \in \mathbb{R}^{n\times m}_+. \tag{2}$$

Definition 2. (*Farina and Rinaldi, 2000; Kaczorek, 2002*) The positive continuous-time system (1) is called *asymptotically stable* if for $u(t) = 0$, $t \geq 0$,

$$\lim_{t\to\infty} x(t) = 0 \quad \text{for all} \quad x_0 \in \mathbb{R}^n_+. \tag{3}$$

Theorem 2. (Farina and Rinaldi, 2000; Kaczorek, 2002) *The positive continuous-time system (1) is asymptotically stable if and only if all coefficients of the polynomial*

$$\det[I_n s - A_c] = s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0 \tag{4}$$

are positive, i.e., $a_i > 0$ *for* $i = 0, 1, \dots, n-1$.

Now let us consider the discrete-time linear system

$$x_{i+1} = A_d x_i + B_d u_i, \quad i \in \mathbb{Z}_+, \tag{5}$$

where $x_i \in \mathbb{R}^n$, $u_i \in \mathbb{R}^m$ are the state and input vectors and $A_d \in \mathbb{R}^{n\times n}$, $B_d \in \mathbb{R}^{n\times m}$.

Definition 3. (*Farina and Rinaldi, 2000; Kaczorek, 2002*) The discrete-time system (5) is called (internally) *positive* if $x_i \in \mathbb{R}^n_+$, $i \in Z_+$, for any initial conditions $x_0 \in \mathbb{R}^n_+$ and all inputs $u_i \in \mathbb{R}^m_+$, $i \in \mathbb{Z}_+$.

Theorem 3. (Farina and Rinaldi, 2000; Kaczorek, 2002) *The discrete-time system (5) is positive if and only if*

$$A_d \in \mathbb{R}^{n\times n}_+, \quad B_d \in \mathbb{R}^{n\times m}_+. \tag{6}$$

Definition 4. (*Farina and Rinaldi, 2000; Kaczorek, 2002*) The positive discrete-time system (5) is called *asymptotically stable* if for $u_i = 0$, $i \in \mathbb{Z}_+$,

$$\lim_{i\to\infty} x_i = 0 \quad \text{for all} \quad x_0 \in \mathbb{R}^n_+. \tag{7}$$

Theorem 4. (Farina and Rinaldi, 2000; Kaczorek, 2002) *The positive discrete-time system (5) is asymptotically stable if and only if all coefficients of the polynomial*

$$\det[I_n(z+1) - A_d] = z^n + \bar{a}_{n-1}z^{n-1} + \cdots + \bar{a}_1 z + \bar{a}_0 \tag{8}$$

are positive, i.e., $\bar{a}_i > 0$ *for* $i = 0, 1, \dots, n-1$.

It is well-known that if sampling is applied to the continuous-time system (1) then the corresponding discrete-time system (5) has the matrices

$$A_d = e^{A_c h}, \quad B_d = \int_0^h e^{A_c t} B_c \,\mathrm{d}t, \tag{9}$$

where $h > 0$ is the sampling period.

If $\det A_c \neq 0$ and $\operatorname{rank} B_c = m$, then from (9) we have

$$B_d = A_c^{-1}(e^{A_c h} - I_n)B_c \tag{10}$$

and

$$\operatorname{rank} B_d = m \tag{11}$$

since $\det[e^{A_c h} - I_n] \neq 0$.

2.2. Fractional positive systems. Consider the continuous-time fractional linear system described by the state equation

$$\frac{\mathrm{d}^\alpha x(t)}{\mathrm{d}t^\alpha} = A_c x(t) + B_c u(t), \quad 0 < \alpha \leq 1, \tag{12}$$

where $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^m$ are respectively the state and input vectors, and $A_c \in \mathbb{R}^{n\times n}$, $B_c \in \mathbb{R}^{n\times m}$,

$$\begin{aligned} \frac{\mathrm{d}^\alpha x(t)}{\mathrm{d}t^\alpha} &= {}_0D_t^\alpha x(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{\dot{x}(\tau)}{(t-\tau)^\alpha} \mathrm{d}\tau, \\ \dot{x}(\tau) &= \frac{\mathrm{d}x(\tau)}{\mathrm{d}\tau} \end{aligned} \tag{13}$$

is the Caputo fractional derivative, while

$$\Gamma(x) = \int_0^\infty e^{-t} t^{x-1} \mathrm{d}t, \quad \operatorname{Re}(x) > 0. \tag{14}$$

is the Euler gamma function.

Theorem 5. (Kaczorek, 2011) *The solution of Eqn. (12) has the form*

$$x(t) = \Phi_0(t)x_0 + \int_0^t \Phi(t-\tau)B_c u(\tau)\,\mathrm{d}\tau, \quad x(0) = x_0 \tag{15}$$

where

$$\Phi_0(t) = \sum_{k=0}^\infty \frac{A_c^k t^{k\alpha}}{\Gamma(k\alpha+1)}, \quad \Phi(t) = \sum_{k=0}^\infty \frac{A_c^k t^{(k+1)\alpha-1}}{\Gamma[(k+1)\alpha]}. \tag{16}$$

Definition 5. The fractional continuous-time system (12) is called (internally) *positive fractional* if the state vector $x(t) \in \mathbb{R}^n_+$, $t \geq 0$, for all initial conditions $x_0 \in \mathbb{R}^n_+$ and all inputs $u(t) \in \mathbb{R}^m_+$, $t \geq 0$.

Theorem 6. (Kaczorek, 2011) *The fractional continuous-time system (12) is internally positive if and only if*

$$A_c \in M_n, \quad B_c \in \mathbb{R}^{n\times m}_+. \tag{17}$$

Theorem 7. (Kaczorek, 2011) *The fractional positive continuous-time system (12) is asymptotically stable if and only if the eigenvalues of A_c are located in the open left half of the complex plane.*

Theorem 8. (Farina and Rinaldi, 2000; Kaczorek, 2002; 2011) *The fractional positive continuous-time system (12) is asymptotically stable if and only if all coefficients of the polynomial (4) are positive, i.e., $a_i > 0$ for $i = 0, 1, \dots, n-1$.*

Now let us consider the fractional discrete-time linear system

$$\Delta^\alpha x_{i+1} = A_d x_i + B_d u_i, \quad i \in \mathbb{Z}_+, \quad 0 < \alpha < 1, \tag{18}$$

where $x_i \in \mathbb{R}^n$ is the state vector, $u_i \in \mathbb{R}^m$ is the input vector and $A_d \in \mathbb{R}^{n\times n}$, $B_d \in \mathbb{R}^{n\times m}$ while

$$\Delta^\alpha x_i = x_i + \sum_{j=1}^{i} (-1)^j \begin{pmatrix} \alpha \\ j \end{pmatrix} x_{i-j}, \begin{pmatrix} \alpha \\ j \end{pmatrix} = \begin{cases} 1 & \text{for} \quad j = 0, \\ \dfrac{\alpha(\alpha-1)\dots(\alpha-j+1)}{j!} & \text{for} \quad j = 1, 2, \dots \end{cases} \tag{19}$$

is the fractional α order difference of x_i.

Substituting (19) into (18) we obtain

$$x_{i+1} = A_\alpha x_i + \sum_{j=2}^{i+1} (-1)^{j+1} \begin{pmatrix} \alpha \\ j \end{pmatrix} x_{i-j+1} + B_d u_i, \quad i \in \mathbb{Z}_+, \tag{20}$$

where

$$A_\alpha = A_d + I_n \alpha. \tag{21}$$

Theorem 9. *The solution of Eqn. (20) has the form*

$$x_i = \Phi_i x_0 + \sum_{j=0}^{i-1} \Phi_{i-j-1} B_d u_i, \tag{22}$$

where the matrix Φ_i can be computed from the formula

$$\Phi_{i+1} = \Phi_i A_\alpha + \sum_{j=2}^{i+1} (-1)^{j+1} \begin{pmatrix} \alpha \\ j \end{pmatrix} \Phi_{i-j+1}, \quad \Phi_0 = I_n. \tag{23}$$

Definition 6. (*Kaczorek, 2011*) The fractional discrete-time system (18) is called (internally) positive if $x_i \in \mathbb{R}_+^n$, $i \in \mathbb{Z}_+$, for all initial conditions $x_0 \in \mathbb{R}_+^n$ and all input sequences $u_i \in \mathbb{R}_+^m$, $i \in \mathbb{Z}_+$.

Theorem 10. (Farina and Rinaldi, 2000; Kaczorek, 2002; 2011) *The fractional discrete-time system (18) is positive if and only if*

$$A_\alpha \in \mathbb{R}_+^{n\times n}, \quad B_d \in \mathbb{R}_+^{n\times m}. \tag{24}$$

Theorem 11. (Kaczorek, 2011) *The fractional positive discrete-time system (18) is asymptotically stable if and only if the positive system*

$$x_{i+1} = (A_d + I_n) x_i \tag{25}$$

is asymptotically stable.

Theorem 12. (Kaczorek, 2011) *The fractional positive discrete-time system (18) is asymptotically stable if and only if all coefficients of the polynomial*

$$\det[I_n z - A_d] = z^n + \hat{a}_{n-1} z^{n-1} + \dots + \hat{a}_1 z + \hat{a}_0 \tag{26}$$

are positive, i.e., $\hat{a}_i > 0$ for $i = 0, 1, \dots, n-1$.

3. Application of a linear Padé approximation

In a similar way as for standard linear systems (Kaczorek, 2013), it can be easily shown that if sampling is applied to the fractional continuous-time system (12) then the corresponding fractional discrete-time system (20) has the matrices

$$A_\alpha = \Phi_0(h), \tag{27a}$$

$$B_d = \int_0^h \Phi(t) B_c \, dt, \tag{27b}$$

where $\Phi_0(t)$ and $\Phi(t)$ are defined by (16).

In this paper the matrix A_d will be approximated by

$$A_d = [A_c + I_n \beta][I_n \beta - A_c]^{-1}, \tag{28a}$$

where the coefficient $\beta > 0$ is chosen so that $A_c + I_n\beta \in \mathbb{R}$ and $[I_n\beta - A_c]^{-1} \in \mathbb{R}_n^{n\times n}$, $h > 0$ is the sampling period. If $A_c \in M_{ns}$, then $\det[I_n\beta - A_c] \in \mathbb{R}_=^{n\times n}$ for any $\beta > 0$.

For $\det A_c \neq 0$, using the Padé approximation from (27b) we obtain

$$B_d = A_c^{-1}\{[A_c + I_n\beta][I_n\beta - A_c]^{-1} - I_n\} B_c. \tag{28b}$$

Remark 1. Knowing A_c, B_c and h and using (27b) we may compute the exact matrix B_d.

Therefore, the fractional continuous-time linear system (12) can be approximated by the fractional discrete-time linear system (20) with the matrices A_d and B_d defined by (27b) and (28), respectively.

Theorem 13. *If the fractional continuous-time system (12) is positive and asymptotically stable, then the corresponding fractional discrete-time system (20) with (27a) and (28b) is also positive and its matrix eigenvalues are located inside the unit circle $|z_k| < 1$ for $k = 1, \dots, n$, and any sampling period $h > 0$.*

Proof. If the fractional continuous-time system (12) is positive and asymptotically stable, then $A_c \in M_{ns}$, and for any $\beta \geq \max(-a^c_{i,i})$ (where $a^c_{i,i}$ is the i-th diagonal entry of A_c, $i = 1, 2, \dots, n$) we have $[A_c + I_n\beta] \in \mathbb{R}^{n\times n}_+$, $[I_n\beta - A_c]^{-1} \in \mathbb{R}^{n\times n}_+$ (Berman and Plemmons, 1994; Kaczorek, 2011) and $A_d \in \mathbb{R}^{n\times n}_+$. From (27b) (and also from (28)) we have $B_d \in \mathbb{R}^{n\times m}_+$ since $\Phi(t) \in \mathbb{R}^{n\times n}_+$, $t \geq 0$ and $B_c \in \mathbb{R}^{n\times m}_+$. Therefore, the corresponding fractional discrete-time system (20) is positive for any $h > 0$ if the fractional continuous-time system (12) is positive and asymptotically stable since (Kaczorek, 2011)

$$(-1)^{j+1}\begin{pmatrix}\alpha\\ j\end{pmatrix} > 0 \tag{29}$$

for $0 < \alpha < 1$ and $j = 2, 3, \dots$.

If the fractional positive system (12) is asymptotically stable, then the real parts α_k of the eigenvalues $s_k = -\alpha_k \pm j\beta_k$, $k = 1, 2, \dots, n$, of the matrix A_c are negative. By Lemma A1 (see Appendix) and Theorem 11, from (A2) we obtain

$$\begin{aligned}|z_k| &= \left|\frac{\beta - \alpha_k \pm j\beta_k}{\beta + \alpha_k \mp j\beta_k}\right| \\ &= \frac{\sqrt{(\beta - \alpha_k)^2 + \beta_k^2}}{\sqrt{(\beta + \alpha_k)^2 + \beta_k^2}} < 1.\end{aligned} \tag{30}$$

Therefore, the eigenvalues of the matrix of the fractional positive discrete-time system (20) with (27a) and (28) are located inside the unit circle for any sampling period $h > 0$. ■

Remark 2. Note that the precision of the approximation of the fractional positive continuous-time system (12) by the fractional positive discrete-time system (20) depends on the choice of the coefficient β. It is recommended to choose the coefficient β so that the square of the difference between the solutions of the continuous-time and discrete-time systems be minimal.

Example 1. Consider the fractional positive and asymptotically stable continuous-time linear system (12) with $0 < \alpha < 1$ and the matrices

$$A_c = \begin{bmatrix} -2 & 1 \\ 0 & -3 \end{bmatrix}, \quad B_c = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \tag{31}$$

The step response of the fractional continuous-time system with matrices (31) and zero initial conditions is given in Fig. 1.

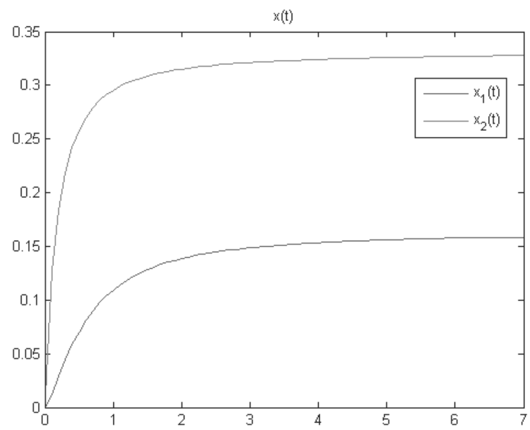

Fig. 1. Step response of the fractional continuous-time system (31).

Using (27a) and (28), we may compute the matrices A_d and B_d of the fractional discrete-time system (20) for $\beta = 4$,

$$\begin{aligned}A_d &= [A_c + I_n\beta][I_n\beta - A_c]^{-1} \\ &= \begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 6 & -1 \\ 0 & 7 \end{bmatrix}^{-1} \\ &= \frac{1}{21}\begin{bmatrix} 7 & 4 \\ 0 & 3 \end{bmatrix},\end{aligned} \tag{32a}$$

$$B_d = \Phi(A_c h) = \begin{bmatrix} 0.0875267 \\ 0.2365546 \end{bmatrix}, \tag{32b}$$

and

$$\begin{aligned}B_d &= A_c^{-1}\{[A_c + I_n\beta][I_n\beta - A_c]^{-1} - I_n\}B_c \\ &= \begin{bmatrix} -2 & 1 \\ 0 & -3 \end{bmatrix}^{-1}\left\{\frac{1}{21}\begin{bmatrix} 7 & 4 \\ 0 & 3 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right\}\begin{bmatrix} 0 \\ 1 \end{bmatrix} \\ &= \frac{1}{21}\begin{bmatrix} 1 \\ 6 \end{bmatrix}.\end{aligned} \tag{33}$$

By Theorem 10, the fractional discrete-time system with (32) is positive.

The step response of the fractional discrete-time system with matrices (32) and zero initial conditions is given in Fig. 2.

The matrix A_c given by (31) of fractional positive asymptotically stable continuous-time systems has the eigenvalues $s_1 = -2$, $s_2 = -3$. Using (A2) we obtain, for $\beta = 4$,

$$\begin{aligned}z_1 &= \frac{\beta + s_1}{\beta - s_1} = \frac{4 - 2}{4 + 2} = \frac{1}{3}, \\ z_2 &= \frac{\beta + s_2}{\beta - s_2} = \frac{4 - 3}{4 + 3} = \frac{1}{7}.\end{aligned} \tag{34}$$

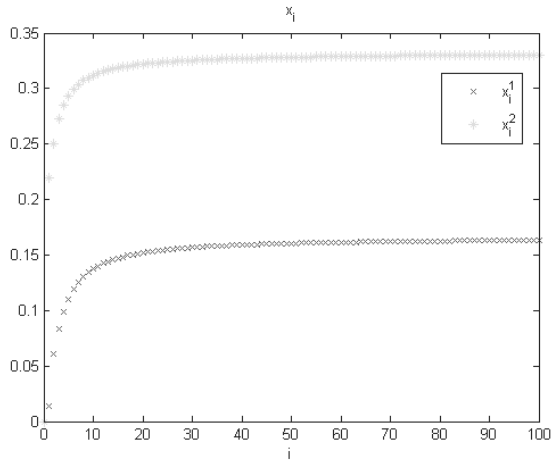

Fig. 2. Step response of the fractional discrete-time system (32).

Therefore, the fractional positive discrete-time system satisfies also the condition $|z_k| < 1$ for $k = 1, \dots, n$. ♦

The discussion of stabilization by state feedbacks of positive systems presented by Kaczorek (2013) can be easily extended to fractional positive linear systems.

4. Concluding remarks

The approximation of fractional positive asymptotically stable continuous-time linear systems with the use of a linear Padé-type approximation by fractional positive asymptotically stable systems was addressed. It was shown that the approximation preserves the positivity and asymptotic stability of the systems (Theorem 13). The optimal choice of the coefficient β so that the square of the difference between solutions of continuous-time and discrete-time systems be minimal was also discussed. The discussion was illustrated with a numerical example. The presented approach can be extended to fractional 2D linear systems (Kaczorek, 2011).

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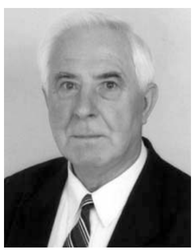

Tadeusz Kaczorek received the M.Sc., Ph.D. and D.Sc. degrees in electrical engineering from the Warsaw University of Technology in 1956, 1962 and 1964, respectively. In the years 1968–69 he was the dean of the Electrical Engineering Faculty, and in the period of 1970–73 he was a deputy rector of the Warsaw University of Technology. In 1971 he became a professor and in 1974 a full professor at the same university. Since 2003 he has been a professor at Białystok Technical University. In 1986 he was elected a corresponding member and in 1996 a full member of the Polish Academy of Sciences. In the years 1988–1991 he was the director of the Research Centre of the Polish Academy of Sciences in Rome. In 2004 he was elected an honorary member of the Hungarian Academy of Sciences. He has been granted honorary doctorates by nine universities. His research interests cover systems theory, especially singular multidimensional systems, positive multidimensional systems, singular positive 1D and 2D systems, as well as positive fractional 1D and 2D systems. He initiated research in the field of singular 2D, positive 2D and positive fractional linear systems. He has published 24 books (six in English) and over 1000 scientific papers. He has also supervised 69 Ph.D. theses. He is the editor-in-chief of the *Bulletin of the Polish Academy of Sciences: Technical Sciences* and a member of editorial boards of ten international journals.

Appendix

Lemma A1. *If s_k, $k = 1, 2, \dots, n$, are eigenvalues of the matrix $A_c \in M_n$, then the eigenvalues z_k, $k = 1, 2, \dots, n$, of the matrix*

$$A_d = [A_c + I_n\beta][I_n\beta - A_c]^{-1} \tag{A1}$$

are given by

$$z_k = \frac{s_k + \beta}{\beta - s_k} \quad \textit{for} \quad k = 1, 2, \dots, n. \tag{A2}$$

Proof. If $A_c \in M_n$, $\beta > 0$ is chosen so that $[A_c + I_n\beta] \in \mathbb{R}_+^{n\times n}$ and $\beta \neq s_k$, then the function

$$f(s_k) = \frac{s_k + \beta}{\beta - s_k}$$

is well defined on the spectrum s_k, $k = 1, 2, \dots, n$, of the matrix A_c. In this case it is well known (Gantmakher, 1959; Kaczorek, 1998; 2013) that the equality (A2) holds. ■

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