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DISCRETIZATION OF SINGULAR SYSTEMS AND ERROR ESTIMATION

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This paper proposes a discretization technique for a descriptor differential system. The methodology used is both triangular first order hold discretization and zero order hold for the input function. Upper bounds for the error between the continuous and the discrete time solution are produced for both discretization methods and are shown to be better than any other existing method in the literature.

Keywords: descriptor systems, discretization, truncation error, first order hold, zero order hold.

1. Introduction

In digital control, and in several areas of engineering, we need to discretize continuous-time state-space equations. The discretization process, though, introduces an error between the continuous and the discretized solution. More specifically, we study Linear Time Invariant (LTI) differential systems of the form

$$E\dot{x}(t) = Ax(t) + Bu(t), \qquad (1)$$

with $E, A \in \mathbb{F}^{n \times n}$, which is the set of all square matrices with elements in the field $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$, and $\det E = 0$ and $B \in \mathbb{F}^{n \times l}$ are constant matrices. We also assume that state vector $x(t) \in \mathbb{F}^{n \times 1}$, where each $x_i(t) : \mathbb{F} \mapsto \mathbb{F}$, has consistent initial conditions and that input vector $u(t) \in \mathbb{F}^{l \times 1}$, where also each $u_i(t) : \mathbb{F} \mapsto \mathbb{F}$.

In the special case where E is invertible and therefore the system is the known state-space system, a zero-order hold discretized model of (1) is given by Levine (2008). A First Order Hold (FOH) discretized model of (1) by extrapolation (resp. interpolation) of the first derivative of the input is given by Toshiyuki and Mituhiko (1993) (resp. Franklin *et al.*, 1997). In the case where E is singular, we may use the forward or backward Euler method, or even the Gear method proposed by Sincovec *et al.* (1981) in order to get a discretized singular model of (1). In the literature on discretization methods for descriptor differential systems, we mainly focus on two different interesting methods. The first one (see Karageorgos *et al.*, 2010; 2011) is based on matrix pencil theory, using the Weierstrass canonical form, and the second one

(see Karampetakis and Gregoriadou, 2011; Karampetakis, 2004; López-Estrada *et al.*, 2012), which is also used is the latest version of Wolfram Mathematica 9, is based on the Laurent expansion of $(sE - A)^{-1}$. Both the methods are somehow equivalent using Zero Order Hold (ZOH) approximation. This paper is an extension to the first method, using triangular first order hold (interpolating FOH) approximation.

Consequently, in this paper, we provide the following interesting results: (a) two new upper bounds for the norm of the difference between the continuous solution and the discretized solution $\|x(kT) - x_k\|$ are given by extending the already known upper bound suggested by Karageorgos *et al.* (2011) for the zero order hold approximation and providing a new upper bound for the first order hold approximation, (b) the proposed bounds penalize our choice for the sampling period T and thus we can estimate a maximum period T if we demand the error to not exceed a given value. Finally, ZOH and interpolating FOH are compared via an example and advantages of interpolating FOH over ZOH are presented.

2. Problem formulation and preliminaries

Linear generalized differential systems of the type $E\dot{x}(t) = Ax(t)$, $E, A \in \mathbb{R}^{n \times n}$ with $\det E = 0$, where $x \in \mathbb{R}^{n \times 1}$ and x_0 is an initial value, are required in the modelling of many physical, electrical and mechanical problems. Systems of this type are related to matrix pencil theory since the algebraic geometric and dynamic properties stem from the structure of the associated pencil

$sE - A$.

Given $E, A \in \mathbb{F}^{m \times n}$ and an indeterminate s, the matrix pencil $sE - A$ is called regular when $m = n$ and $\det(sE - A) \neq 0$. In any other case, the pencil will be called singular. The pencil $sE - A$ is said to be strictly equivalent to the pencil $s\tilde{E} - \tilde{A}$ if and only if there exist $P, Q \in \mathbb{C}^{n \times n}$ such that $P(sE - A)Q = s\tilde{E} - \tilde{A}$, where $\det P, \det Q \neq 0$. It is known (Gantmacher, 1959) that $sE - A$ is strictly equivalent to its Weierstrass normal form $sE_w - A_w$, i.e., there exist nonsingular matrices P, Q such that

$$P(sE - A)Q = \begin{pmatrix} sI_p - J_p & 0 \\ 0 & sH_q - I_q \end{pmatrix}$$
$$= sE_w - A_w,$$

where $H_q \in \mathbb{R}^{q \times q}$ is nilpotent and $J_p \in \mathbb{R}^{p \times p}$ with $p + q = n$,

$$H_q = \text{blockdiag}\{H_{q_1}, \ldots, H_{q_k}\},$$

$$H_{q_i} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix} \in \mathbb{R}^{\mu_i \times \mu_i},$$

$i \leq k$ with $\sum_{i=1}^{k} \mu_i = q$,

$$J_p = \text{blockdiag}\{J_{\sigma_1}(a_1), \ldots, J_{\sigma_\ell}(a_\ell)\},$$

$$J_{\sigma_i}(a_i) = \begin{bmatrix} a_i & 1 & 0 & \cdots & 0 \\ 0 & a_i & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & a_i \end{bmatrix} \in \mathbb{R}^{\sigma_i \times \sigma_i},$$

$i \leq \ell$, with $\sum_{i=1}^{\ell} \sigma_i = p$. Here $\ell \geq 0$ is the number of the *finite elementary divisors* (f.e.d.) of $sE - A$ of the form $(s - a_i)^{\sigma_i}$ which uniquely characterize the block $sI_p - J_p$. The infinite elementary divisors (i.e.d.) of $sE - A$, which uniquely characterize the block $sH_q - I_q$, are given by

$$w^{\mu_1}, w^{\mu_2}, \ldots, w^{\mu_k},$$

where μ_i are the sizes of the Jordan blocks $H_{q_i}, i \in k$, of H_q and they can be defined as the f.e.d.'s of the "dual" pencil $E - wA$ at $w = 0$. The relation between the i.e.d. and the infinite pole-zero structure of $sE - A$ is given by Vardulakis and Karcanias (1983). The matrices P, Q used for transforming $sE - A$ to $sE_w - A_w$ are not unique. A numerical algorithm is given by Duan (2010) for the calculation of these matrices, whereas a theoretical algorithm based on the finite and infinite generalized eigenvectors of the matrix pencil $sE - A$ is given by Vardulakis (1991).

Now, we consider the transformation $x(t) = Qy(t)$ and obtain the following results. As it has been already mentioned about the mathematical tools used during the discretization process, only the Weierstrass Canonical Form (WCF) is required. As this paper extends the work of Karageorgos *et al.* (2010) using first order hold approximation instead of zero order hold in order to get better results, some commonly used lemmas are presented without their proofs, although full references are provided. We already know that the system (1) has the following continuous time solution (see Dai, 1989; Karageorgos *et al.*, 2010; Koumboulis and Mertzios, 1999):

$$x(t)$$
$$= Q_{n,p}\left(e^{J_p(t-t_0)}y_p(t_0) + \int_{t_0}^{t} e^{J_p(t-s)}B_{p,l}u(s)\,\mathrm{d}s\right)$$
$$- Q_{n,q}\sum_{i=0}^{q^*-1} H_q^i B_{q,l}u^{(i)}(t), \tag{2}$$

where

$$Q = \begin{bmatrix} Q_{n,p} & Q_{n,q} \end{bmatrix}, \quad B = \begin{bmatrix} B_{p,l} \\ B_{q,l} \end{bmatrix},$$

$$y(t_0) = \begin{bmatrix} y_p(t_0) \\ y_q(t_0) \end{bmatrix} = Q^{-1}x(t_0)$$

and $u^{(i)}(t)$ is the i-th derivative of the input function $u(t)$. However, (2) can be transformed in a more useful format.

We have

$$x(t)$$
$$= Q_{n,p}e^{J_p(t-t_0)}y_p(t_0) + Q_{n,q}y_q(t_0)$$
$$+ Q_{n,p}\int_{t_0}^{t} e^{J_p(t-s)}B_{p,l}u(s)\,\mathrm{d}s - Q_{n,q}y_q(t_0)$$
$$- Q_{n,q}\sum_{i=0}^{q^*-1} H_q^i B_{q,l}u^{(i)}(t)$$
$$= \begin{bmatrix} Q_{n,p} & Q_{n,q} \end{bmatrix} \begin{bmatrix} e^{J_p(t-t_0)} & O_{p,q} \\ O_{q,p} & I_q \end{bmatrix} \begin{bmatrix} y_p(t_0) \\ y_q(t_0) \end{bmatrix}$$
$$+ Q_{n,p}\int_{t_0}^{t} e^{J_p(t-s)}B_{p,l}u(s)\,\mathrm{d}s$$
$$+ Q_{n,q}\left(-y_q(t_0) - \sum_{i=0}^{q^*-1} H_q^i B_{q,l}u^{(i)}(t)\right).$$

In order to obtain consistent initial conditions for the system (1) (see Karageorgos *et al.*, 2010), we should consider that

$$\begin{bmatrix} y_p(t_0) \\ y_q(t_0) \end{bmatrix} = Q^{-1}x(t_0),$$

$$-y_q(t_0) = \sum_{i=0}^{q^*-1} H_q^i B_{q,l}u^{(i)}(t_0)),$$

and as a result we obtain

$$x(t) = Q \begin{bmatrix} e^{J_p(t-t_0)} & O_{p,q} \\ O_{q,p} & I_q \end{bmatrix} Q^{-1} x(t_0)$$

$$+ Q_{n,p} \int_{t_0}^t e^{J_p(t-s)} B_{p,l} u(s) \, \mathrm{d}s$$

$$+ Q_{n,q} \sum_{i=0}^{q^*-1} H_q^i B_{q,l} \left(u^{(i)}(t_0) - u^{(i)}(t) \right).$$

Moreover, by definition, the state-transition matrix of the autonomous linear descriptor differential system, $E\dot{x}(t) = Ax(t)$, is given by

$$\Phi(t,t_0) = Q \begin{bmatrix} e^{J_p(t-t_0)} & O_{p,q} \\ O_{q,p} & I_q \end{bmatrix} Q^{-1}.$$

Finally, after noticing that

$$\Phi(t,s) Q_{n,p} = \Phi(t,s) \begin{bmatrix} Q_{n,p} & Q_{n,q} \end{bmatrix} \begin{bmatrix} I_{p,p} \\ O_{q,p} \end{bmatrix}$$

$$= \begin{bmatrix} Q_{n,p} e^{J_p(t-s)} & Q_{n,q} \end{bmatrix} \begin{bmatrix} I_{p,p} \\ O_{q,p} \end{bmatrix}$$

$$= Q_{n,p} e^{J_p(t-s)},$$

we get

$$x(t) = \Phi(t,t_0) x(t_0) + \int_{t_0}^t \Phi(t,s) Q_{n,p} B_{p,l} u(s) \, \mathrm{d}s$$

$$+ Q_{n,q} \sum_{i=0}^{q^*-1} H_q^i B_{q,l} \left(u^{(i)}(t_0) - u^{(i)}(t) \right). \quad (3)$$

Now, let $T > 0$ be a constant sampling period. We also assume that $t_0 = 0$. We consider two cases. In the first one, the input function $u(\tau)$ is constant in the interval $[kT, (k+1)T)$ and we approximate it by using ZOH approximation,

$$u(\tau) = u(kT), \qquad \forall \tau \in [kT, (k+1)T).$$

In the second case, the input function $u(\tau)$ is not constant in the interval $[kT, (k+1)T)$ and we approximate it by using triangular first order hold (interpolating FOH) approximation,

$$u(\tau) = u(kT) + \frac{u((k+1)T) - u(kT)}{T} (\tau - kT),$$

$\forall \tau \in [kT, (k+1)T)$. In order to combine these formulas into one, we write

$$u(\tau) = u(kT) + \chi_{tf} \frac{u((k+1)T) - u(kT)}{T} (\tau - kT),$$

$\forall \tau \in [kT, (k+1)T)$, where $\chi_{tf} = 1$ or 0 depending on whether we consider interpolating FOH or ZOH approximation, respectively. For simplicity, hereafter, we use the notation $x_k := x(kT), \forall k = 0, 1, 2, \ldots$. From Eqn. (3), by setting $t = kT$ and $t = (k+1)T$, we get

$$x_k = \Phi(kT, 0) x_0 + Q_{n,q} \sum_{i=0}^{q^*-1} H_q^i B_{q,l}(u_0^{(i)} - u_k^{(i)})$$

$$+ \int_0^{kT} \Phi(kT, s) Q_{n,p} B_{p,l} u(s) \, \mathrm{d}s, \qquad (4)$$

$$x_{k+1} = \Phi((k+1)T, 0) x_0$$

$$+ Q_{n,q} \sum_{i=0}^{q^*-1} H_q^i B_{q,l}(u_0^{(i)} - u_{k+1}^{(i)})$$

$$+ \int_0^{(k+1)T} \Phi((k+1)T, s) Q_{n,p} B_{p,l} u(s) \, \mathrm{d}s. \qquad (5)$$

Based on the group property of the flow, we arrive at the following lemma.

Lemma 1. *The following equalities hold:*

$$\Phi(T, 0)\Phi(kT, s) = \Phi((k+1)T, s),$$
$$\Phi(T, 0)Q_{n,q} = Q_{n,q}.$$

From Eqns. (4) and (5) and using the above lemma, we multiply x_k by $\Phi(T, 0)$ and then subtract from x_{k+1} to finally get

$$x_{k+1} - \Phi(T, 0) x_k$$

$$= Q_{n,q} \sum_{i=0}^{q^*-1} H_q^i B_{q,l}((u_0^{(i)} - u_{k+1}^{(i)})$$

$$- \underbrace{\Phi(T, 0) Q_{n,q}}_{\mathbf{Q_{n,q}}} \sum_{i=0}^{q^*-1} H_q^i B_{q,l}(u_0^{(i)} - u_k^{(i)})$$

$$+ \int_{kT}^{(k+1)T} \Phi((k+1)T, s) Q_{n,p} B_{p,l} u(s) \, \mathrm{d}s,$$

and therefore the following recursive formula is derived:

$$x_{k+1} = \Phi(T, 0) x_k$$

$$+ Q_{n,q} \sum_{i=0}^{q^*-1} H_q^i B_{q,l}(u_k^{(i)} - u_{k+1}^{(i)})$$

$$+ \int_{kT}^{(k+1)T} \Phi((k+1)T, s) Q_{n,p} B_{p,l} u(s) \, \mathrm{d}s. \qquad (6)$$

But

$$\int_{kT}^{(k+1)T} \Phi((k+1)T, s)Q_{n,p}B_{p,l}u(s)\,\mathrm{d}s$$

$$\overset{s=kT+w}{=} \int_0^T \Phi((k+1)T, kT+w)Q_{n,p}$$

$$\times B_{p,l}u(kT+w)\,\mathrm{d}w$$

$$= \int_0^T \Phi(T-w, 0)Q_{n,p}$$

$$\times B_{p,l}\left(u_k + \chi_{tf}\frac{u_{k+1}-u_k}{T}w\right)\mathrm{d}w$$

$$= \int_0^T \Phi(T-w, 0)Q_{n,p}B_{p,l}u_k\,\mathrm{d}w$$

$$+ \chi_{tf}\int_0^T \Phi(T-w, 0)Q_{n,p}$$

$$\times B_{p,l}\frac{u_{k+1}-u_k}{T}w\,\mathrm{d}w. \tag{7}$$

Finally, by setting $\lambda = T - w$ in (7) and replacing in (6), we get the following recursive formula:

$$x_{k+1} = \Phi(T,0)x_k + Q_{n,q}\sum_{i=0}^{q^*-1} H_q^i B_{q,l}(u_k^{(i)} - u_{k+1}^{(i)})$$

$$+ \int_0^T \Phi(\lambda, 0)\,\mathrm{d}\lambda Q_{n,p}B_{p,l}u_k$$

$$+ \chi_{tf}\int_0^T \Phi(\lambda, 0)(T-\lambda)\,\mathrm{d}\lambda$$

$$\times Q_{n,p}B_{p,l}\frac{u_{k+1}-u_k}{T}. \tag{8}$$

The relation (8) is the discretized model of (1) under ZOH or interpolating FOH approximation.

Theorem 1. *The solution of (3) under interpolating FOH ($\chi_{tf} = 1$) or ZOH ($\chi_{tf} = 0$) approximation is given by the following analytic formula:*

$$x_k$$

$$= \Phi(kT, 0)x_0 + Q_{n,q}\sum_{i=0}^{q^*-1} H_q^i B_{q,l}(u_0^{(i)} - u_k^{(i)})$$

$$+ \sum_{j=0}^{k-1}\int_0^T \Phi(jT+\lambda, 0)\,\mathrm{d}\lambda Q_{n,p}B_{p,l}u_{k-j-1}$$

$$+ \chi_{tf}\sum_{j=0}^{k-1}\int_0^T \Phi(jT+\lambda, 0)(T-\lambda)\,\mathrm{d}\lambda\, Q_{n,p}B_{p,l}$$

$$\times \frac{u_{k-j}-u_{k-j-1}}{T}. \tag{9}$$

Proof. First of all, for $k = 0$ in (8) we have the case $k = 1$ in (9). We assume that this is true for $k - 1$, that is,

$$x_{k-1} = \Phi((k-1)T, 0)x_0$$

$$+ Q_{n,q}\sum_{i=0}^{q^*-1} H_q^i B_{q,l}(u_0^{(i)} - u_{k-1}^{(i)})$$

$$+ \sum_{j=0}^{k-2}\int_0^T \Phi(jT+\lambda, 0)\,\mathrm{d}\lambda Q_{n,p}B_{p,l}u_{k-j-2}$$

$$+ \chi_{tf}\sum_{j=0}^{k-2}\int_0^T \Phi(jT+\lambda, 0)(T-\lambda)\,\mathrm{d}\lambda Q_{n,p}B_{p,l}$$

$$\times \frac{u_{k-j-1}-u_{k-j-2}}{T},$$

and we prove it for k. By replacing x_{k-1} in the recursive formula (8), we get

$$x_k$$

$$= \Phi(T,0)\Bigg(\Phi((k-1)T, 0)x_0$$

$$+ Q_{n,q}\sum_{i=0}^{q^*-1} H_q^i B_{q,l}(u_0^{(i)} - u_{k-1}^{(i)})$$

$$+ \sum_{j=0}^{k-2}\int_0^T \Phi(jT+\lambda, 0)\,\mathrm{d}\lambda\, Q_{n,p}B_{p,l}u_{k-j-2}$$

$$+ \chi_{tf}\sum_{j=0}^{k-2}\int_0^T \Phi(jT+\lambda, 0)(T-\lambda)\,\mathrm{d}\lambda\, Q_{n,p}B_{p,l}$$

$$\times \frac{u_{k-j-1}-u_{k-j-2}}{T}\Bigg)$$

$$+ \int_0^T \Phi(\lambda, 0)\,\mathrm{d}\lambda\, Q_{n,p}B_{p,l}u_{k-1}$$

$$+ \chi_{tf}\int_0^T \Phi(\lambda, 0)(T-\lambda)\,\mathrm{d}\lambda\, Q_{n,p}B_{p,l}\frac{u_k - u_{k-1}}{T}$$

$$+ Q_{n,q}\sum_{i=0}^{q^*-1} H_q^i B_{q,l}(u_{k-1}^{(i)} - u_k^{(i)})$$

or, equivalently,

$$x_k$$

$$= \Phi(kT, 0)x_0 + \underbrace{\Phi(T,0)Q_{n,q}}_{Q_{n,q}}\sum_{i=0}^{q^*-1} H_q^i B_{q,l}(u_0^{(i)} - u_{k-1}^{(i)})$$

$$+ Q_{n,q}\sum_{i=0}^{q^*-1} H_q^i B_{q,l}(u_{k-1}^{(i)} - u_k^{(i)})$$

$$+ \sum_{j=0}^{k-2}\int_0^T \Phi((j+1)T+\lambda, 0)\,\mathrm{d}\lambda Q_{n,p}B_{p,l}u_{k-j-2}$$

$$+ \int_0^T \Phi(\lambda, 0) \, d\lambda \, Q_{n,p} B_{p,l} u_{k-1}$$

$$+ \chi_{tf} \sum_{j=0}^{k-2} \int_0^T \Phi((j+1)T + \lambda, 0)(T - \lambda) \, d\lambda$$

$$\times Q_{n,p} B_{p,l} \frac{u_{k-j-1} - u_{k-j-2}}{T}$$

$$+ \chi_{tf} \int_0^T \Phi(\lambda, 0)(T - \lambda) \, d\lambda \, Q_{n,p} B_{p,l} \frac{u_k - u_{k-1}}{T}.$$

Now, by setting $i = j + 1$ in order to group similar terms, we have

$$x_k$$
$$= \Phi(kT, 0)x_0$$

$$+ Q_{n,q} \sum_{i=0}^{q^*-1} H_q^i B_{q,l} (u_0^{(i)} - u_{k-1}^{(i)} + u_{k-1}^{(i)} - u_k^{(i)})$$

$$+ \sum_{i=1}^{k-1} \int_0^T \Phi(iT + \lambda, 0) \, d\lambda \, Q_{n,p} B_{p,l} u_{k-i-1}$$

$$+ \int_0^T \Phi(\lambda, 0) \, d\lambda \, Q_{n,p} B_{p,l} u_{k-1}$$

$$+ \chi_{tf} \sum_{i=1}^{k-1} \int_0^T \Phi(iT + \lambda, 0)(T - \lambda) \, d\lambda \, Q_{n,p} B_{p,l}$$

$$\times \frac{u_{k-i} - u_{k-i-1}}{T}$$

$$+ \chi_{tf} \int_0^T \Phi(\lambda, 0)(T - \lambda) \, d\lambda$$

$$\times Q_{n,p} B_{p,l} \frac{u_k - u_{k-1}}{T},$$

which completes the induction. ∎

3. Error analysis and upper bound

Having already found an analytic formula for the discretized solution x_k, we provide an analytic expression for the norm of the difference between the continuous time solution at the moments $t = kT$ and the discrete points x_k of the discretized solution. Moreover, we bound this norm and we end up with two upper bounds for ZOH and interpolating FOH, respectively. From (3) and (9), we get

$$x(kT) - x_k$$

$$= \int_0^{kT} \Phi(kT, s) Q_{n,p} B_{p,l} u(s) \, ds$$

$$- \sum_{j=0}^{k-1} \int_0^T \Phi(jT + \lambda, 0) Q_{n,p} B_{p,l}$$

$$\times \left(u_{k-j-1} + \chi_{tf}(T - \lambda) \frac{u_{k-j} - u_{k-j-1}}{T} \right) d\lambda$$

or, by making the substitution $T - \lambda = w$,

$$x(kT) - x_k$$

$$= \int_0^{kT} \Phi(kT, s) Q_{n,p} B_{p,l} u(s) \, ds$$

$$- \sum_{j=0}^{k-1} \int_0^T \Phi((j+1)T - w, 0) Q_{n,p} B_{p,l}$$

$$\times \left(u_{k-j-1} + \chi_{tf} w \frac{u_{k-j} - u_{k-j-1}}{T} \right) dw$$

$$= \sum_{j=0}^{k-1} \int_{jT}^{(j+1)T} \Phi(kT, s) Q_{n,p} B_{p,l} u(s) \, ds$$

$$- \sum_{j=0}^{k-1} \int_0^T \Phi((j+1)T - w, 0) Q_{n,p} B_{p,l}$$

$$\times \left(u_{k-j-1} + \chi_{tf} w \frac{u_{k-j} - u_{k-j-1}}{T} \right) dw.$$

By setting $i = k - j - 1$, we get

$$x(kT) - x_k$$

$$= \sum_{j=0}^{k-1} \int_{jT}^{(j+1)T} \Phi(kT, s) Q_{n,p} B_{p,l} u(s) \, ds$$

$$- \sum_{i=0}^{k-1} \int_0^T \Phi((k-i)T - w, 0) Q_{n,p} B_{p,l}$$

$$\times \left(u_i + \chi_{tf} w \frac{u_{i+1} - u_i}{T} \right) dw$$

$$\overset{j=i}{=} \sum_{i=0}^{k-1} \int_{iT}^{(i+1)T} \Phi(kT, s) Q_{n,p} B_{p,l} u(s) \, ds$$

$$- \sum_{i=0}^{k-1} \int_0^T \Phi((k-i)T - w, 0) Q_{n,p} B_{p,l}$$

$$\times \left(u_i + \chi_{tf} w \frac{u_{i+1} - u_i}{T} \right) dw.$$

We now set $\lambda = w + iT$ and have

$$x(kT) - x_k$$

$$= \sum_{i=0}^{k-1} \int_{iT}^{(i+1)T} \Phi(kT, s) Q_{n,p} B_{p,l} u(s) \, ds$$

$$- \sum_{i=0}^{k-1} \int_{iT}^{(i+1)T} \Phi(kT - \lambda, 0) Q_{n,p} B_{p,l}$$

$$\times \left(u_i + \chi_{tf}(\lambda - iT) \frac{u_{i+1} - u_i}{T} \right) d\lambda$$

$$\overset{\lambda \equiv s}{=} \sum_{i=0}^{k-1} \int_{iT}^{(i+1)T} \Phi(kT,s)Q_{n,p}B_{p,l}u(s)\,\mathrm{d}s$$

$$- \sum_{i=0}^{k-1} \int_{iT}^{(i+1)T} \Phi(kT,s)Q_{n,p}B_{p,l}$$

$$\times \left(u_i + \chi_{tf}(s-iT)\frac{u_{i+1}-u_i}{T} \right)\mathrm{d}s.$$

Thus, finally, we have

$$x(kT) - x_k$$
$$= \sum_{i=0}^{k-1} \int_{iT}^{(i+1)T} \Phi(kT,s)Q_{n,p}B_{p,l}$$
$$\times \left(u(s) - u_i - \chi_{tf}(s-iT)\frac{u_{i+1}-u_i}{T} \right)\mathrm{d}s. \quad (10)$$

Having now in compact form the difference between the continuous and the discretized solution, we have the following interesting results.

Theorem 2. *The upper bound of the error of (3) under ZOH ($\chi_{tf} = 0$) approximation is given by*

$$\|x(kT) - x_k\|$$
$$\leq M_1\|Q_{n,p}\|\|B_{p,l}\|\|Q\|\|Q^{-1}\|$$
$$\times \left\{ \frac{\left(e^{\|J_p\|T} - \|J_p\|T - 1\right)\left(e^{\|J_p\|kT}-1\right)}{\|J_p\|^2\left(e^{\|J_p\|T}-1\right)} \right.$$
$$\left. + \sqrt{q}\frac{kT^2}{2} \right\}, \quad (11)$$

while under interpolating FOH ($\chi_{tf} = 1$) approximation it is given by

$$\|x(kT) - x_k\|$$
$$\leq \frac{1}{8}M_2T^2\|Q_{n,p}\|\|B_{p,l}\|$$
$$\times \|Q\|\|Q^{-1}\|\left\{ \frac{e^{\|J_p\|kT}-1}{\|J_p\|} + kT\sqrt{q} \right\}. \quad (12)$$

Proof. For ZOH approximation ($\chi_{tf} = 0$), we get

$$\|x(kT) - x_k\|$$
$$\leq \|Q_{n,p}\|\|B_{p,l}\|$$
$$\times \sum_{i=0}^{k-1} \int_{iT}^{iT+T} \|\Phi(kT,s)\|\|u(s)-u_i\|\,\mathrm{d}s.$$

But from Theorem 12.2.3 of Davidson and Donsig (2010), we have that

$$\|u(s) - u_i\| \leq (s-iT)\|u'(c)\|$$

with $c \in (iT, iT+T)$. Also, we have that

$$\|\Phi(kT,s)\| = \|Q\begin{bmatrix} e^{J_p(kT-s)} & O_{p,q} \\ O_{q,p} & I_q \end{bmatrix}Q^{-1}\|$$
$$\leq \|Q\|\left\{ \|e^{J_p(kT-s)}\| + \sqrt{q} \right\}\|Q^{-1}\|$$

and $\|e^{J_p kT}\| \leq e^{\|J_p\|kT}$, and so we finally get

$$\|x(kT) - x_k\|$$
$$\leq M_1\|Q_{n,p}\|\|B_{p,l}\|\|Q\|e^{\|J_p\|kT}\|Q^{-1}\|$$
$$\times \sum_{i=0}^{k-1} \int_{iT}^{iT+T} e^{-\|J_p\|s}(s-iT)\,\mathrm{d}s$$
$$+ M_1\|Q_{n,p}\|\|B_{p,l}\|\|Q\|\|Q^{-1}\|\sqrt{q}$$
$$\times \sum_{i=0}^{k-1} \int_{iT}^{iT+T} (s-iT)\,\mathrm{d}s,$$

where $M_k = \|u^{(k)}(t)\|_\infty$, $t \in [0, kT]$. By doing some calculations, we get

$$\sum_{i=0}^{k-1} \int_{iT}^{iT+T} e^{-\|J_p\|s}(s-iT)\,\mathrm{d}s$$
$$= \frac{e^{\|J_p\|T} - \|J_p\|T - 1}{\|J_p\|^2} \cdot \frac{1 - e^{-\|J_p\|kT}}{e^{\|J_p\|T}-1}$$

and

$$\sum_{i=0}^{k-1} \int_{iT}^{iT+T} (s-iT)\,\mathrm{d}s = k\frac{T^2}{2},$$

and, finally, the upper bound formula for ZOH is

$$\|x(kT) - x_k\|$$
$$\leq M_1\|Q_{n,p}\|\|B_{p,l}\|\|Q\|\|Q^{-1}\|$$
$$\times \left\{ \frac{\left(e^{\|J_p\|T} - \|J_p\|T - 1\right)\left(e^{\|J_p\|kT}-1\right)}{\|J_p\|^2\left(e^{\|J_p\|T}-1\right)} \right.$$
$$\left. + \sqrt{q}\frac{kT^2}{2} \right\}.$$

Now for interpolating FOH approximation ($\chi_{tf} = 1$), we have that

$$\|x(kT) - x_k\|$$
$$\leq \|Q_{n,p}\|\|B_{p,l}\|\left(\sum_{i=0}^{k-1} \int_{iT}^{(i+1)T} \|\Phi(kT,s)\| \right.$$
$$\times \|u(s) - u_i - (s-iT)\frac{u_{i+1}-u_i}{T}\|\,\mathrm{d}s \Bigg).$$

The polynomial $u_i + (s-iT)\frac{u_{i+1}-u_i}{T}$ interpolates the function $u(s)$, and so

$$\|u(s) - u_i - (s-iT)\frac{u_{i+1}-u_i}{T}\|$$
$$\leq \frac{1}{4(n+1)}M_2\left(\frac{b-a}{n}\right)^{n+1} = \frac{1}{8}M_2T^2$$

because $n = 1$ and $b - a = (iT + T) - iT = T$. At this point, we have

$$\|x(kT) - x_k\|$$
$$\leq \frac{1}{8} M_2 T^2 \|Q_{n,p}\| \|B_{p,l}\| \sum_{i=0}^{k-1} \int_{iT}^{(i+1)T} \|\Phi(kT, s)\| \, \mathrm{d}s.$$

Finally, because

$$e^{\|J_p\| kT} \sum_{i=0}^{k-1} \int_{iT}^{(i+1)T} e^{-\|J_p\| s} \mathrm{d}s$$
$$= e^{\|J_p\| kT} \sum_{i=0}^{k-1} \left(\frac{e^{-\|J_p\| iT}}{\|J_p\|} - \frac{e^{-\|J_p\|(iT+T)}}{\|J_p\|} \right)$$
$$= \frac{e^{\|J_p\| kT} - 1}{\|J_p\|},$$

we get that the upper bound for interpolating FOH is,

$$\|x(kT) - x_k\|$$
$$\leq \frac{1}{8} M_2 T^2 \|Q_{n,p}\| \|B_{p,l}\| \|Q\| \|Q^{-1}$$
$$\times \| \left\{ \frac{e^{\|J_p\| kT} - 1}{\|J_p\|} + kT\sqrt{q} \right\}.$$

$\blacksquare$

The formulas (11) and (12), for ZOH and interpolating FOH, respectively, are the upper bounds we wanted to prove.

The difference of these two formulas from the respective formulas of Karageorgos *et al.* (2010; 2011) is the result of two factors. Firstly, the discretization of the input function $u(t)$ used in this paper is not only zero order hold approximation but, in addition to this, we are also using triangular first order hold discretization. Secondly, a sharp upper bound for $\|\Phi(kT, s)\|$, which appears in both the cases (ZOH and interpolating FOH), contributes to a better general result. Now, we can proceed to the comparison throughout an example.

4. Illustrative example

Let us now consider a system of the form $E\dot{x}(t) = Ax(t) + Bu(t)$, that is,

$$\begin{bmatrix} -1.5 & 2 & 1.5 & 0.5 \\ 0.5 & 0 & -0.5 & -0.5 \\ 0.5 & -1 & -0.5 & 0.5 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \dot{x}_3(t) \\ \dot{x}_4(t) \end{bmatrix}$$
$$= \begin{bmatrix} 0 & 0 & -1 & 1 \\ 0.5 & 0 & -0.5 & -0.5 \\ -0.5 & 1 & 1.5 & -0.5 \\ 0.5 & -1 & -0.5 & 0.5 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 2 \\ 1 \\ 1 \end{bmatrix} \begin{bmatrix} u(t) \end{bmatrix}.$$

Then there exist nonsingular matrices

$$P = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad Q = \begin{bmatrix} 1 & 2 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}$$

such that

$$PEQ = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad PAQ = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Since there are not unique Q, P that transform $sE - A$ to $sE_w - A_w$ and the error depends on Q, we may select the one with the least norm. However, we do not have to proceed with such details. For this system we have $p = q = 2$, $n = 4$. Assume also that $u(t) = t^3$, $k = 500$ and $T = 10^{-3}$. As a result, $M_1 = \|u^{(1)}(t)\|_\infty = 3/4$ and $M_2 = \|u^{(2)}(t)\|_\infty = 3$ with $t \in [0, kT]$. Moreover,

$$Q_{4,2} = \begin{bmatrix} 1 & 2 \\ 1 & 1 \\ 0 & 0 \\ 1 & 0 \end{bmatrix}, \quad B_{2,1} = \begin{bmatrix} 0 \\ 2 \end{bmatrix}.$$

Therefore, $\|Q\| = 2\sqrt{3}$, $\|Q^{-1}\| = \sqrt{21}/2$, $\|Q_{4,2}\| = \sqrt{8}$ and $\|B_{2,1}\| = 2$. Applying these values to the formulas (11) and (12), we get that the upper bound for ZOH is 0.02529229 while for interpolating FOH it is 2.529615×10^{-5}, about 10^{-3} times smaller.

Also, we can estimate the maximum allowed sampling period for which the error does not exceed a given value. For instance, if we want the error not to exceed 10^{-2} for $k = 100$, for ZOH we get $T_{\max} = 0.00153203$ while for interpolating FOH $T_{\max} = 0.0110291$. This proves the fact that, due to the better approximation that interpolating FOH offers instead of ZOH, we do not need to sample our system so often in order to get it under the maximum error allowed.

The last thing to do is to compare these two upper bounds as steps (k) increase. Table 1 shows the values of the upper bounds for $T = 10^{-3}$. From this table we can see that, although for small k ZOH is quite good, when k increases interpolating FOH is significantly better.

5. Conclusion

In this paper, new upper bound formulas regarding the discretization error of a singular descriptor system are considered. These two bounds differ on the way we approximate the input function, either zero order hold or triangular first order hold (interpolating FOH). In addition to this, the improvements of these sharper bounds stem from the upper bound of $\|\Phi(kT, s)\|$ which yields

Table 1. Comparison of upper bounds.

k	ZOH	FOH
1	4.0659×10^{-5}	4.0664×10^{-8}
2	8.1347×10^{-5}	8.1357×10^{-8}
3	1.2206×10^{-4}	1.2208×10^{-7}
4	1.6281×10^{-4}	1.6283×10^{-7}
5	2.0359×10^{-4}	2.0361×10^{-7}
10	4.0791×10^{-4}	4.0796×10^{-7}
100	0.0042190	4.2195×10^{-6}
500	0.025292	2.5296×10^{-5}
750	0.043766	4.3773×10^{-5}
1000	0.069024	6.9037×10^{-5}

a better overall result than that which was proposed by Karageorgos *et al.* (2011). The whole theory is illustrated by an example. The results presented in this work and by Karageorgos *et al.* (2011; 2010) can be further extended to descriptor systems with delay (Jugo, 2002; Chen and Wang, 1999), descriptor fractional systems (Kaczorek, 2013) or even more to autoregressive moving average representations. Alternatively, we can use the fundamental matrix sequence of the matrix pencil $sE - A$, in order to extend the results presented by Karampetakis and Gregoriadou (2011) to the triangular first order hold method and compare with the existing results of this work. Instead of the Weierstrass canonical form, other canonical forms can also be used like the ones presented by Kaczorek (2003). Other hold methods can also be applied, e.g., the first order hold method (backward-Euler approximation of the derivative of the input) that can be combined with several hold methods for the approximation of the derivative of the inputs.

Instead of studying the use of zero order hold devices, we can also study, with the same approach that we employ in this work, the use of fractional order hold devices (or generalized first order (Jury, 1958)) that can improve, if properly tuned, the performance of hybrid control systems (Basterretxea *et al.*, 2008).

Acknowledgment

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