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AN UNCONDITIONALLY POSITIVE AND GLOBAL STABILITY PRESERVING NSFD SCHEME FOR AN EPIDEMIC MODEL WITH VACCINATION

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In this paper, a NonStandard Finite Difference (NSFD) scheme is constructed, which can be used to determine numerical solutions for an epidemic model with vaccination. Here the NSFD method is employed to derive a set of difference equations for the epidemic model with vaccination. We show that difference equations have the same dynamics as the original differential system, such as the positivity of the solutions and the stability of the equilibria, without being restricted by the time step. Our proof of global stability utilizes the method of Lyapunov functions. Numerical simulation illustrates the effectiveness of our results.

Keywords: nonstandard finite differences, unconditional positivity, stability, Lyapunov function.

1. Introduction

The application of theories of Differential Equations (DEs) in mathematical epidemiology has spread rapidly. Various epidemic models, which describe the dynamical behavior of the transmission of infectious disease, have been proposed in the literature (Enszer and Stadtherr, 2009; Ma *et al.*, 2004; Kouche and Ainseba, 2010). The solutions of DEs describe the evolution of different classes of subpopulations in the model for different times. Therefore, one important task of mathematical modeling is to obtain accurate theoretical solutions (Jódar *et al.*, 2008). However, theoretical solutions of these systems exist only occasionally and are usually complicated, so good approximations are necessary (Parra *et al.*, 2010). Numerical methods are often the method of choice (Sekiguchi, 2009; Jang, 2007; Chinviriyasit and Chinviriyasit, 2010; Enatsu *et al.*, 2010; Muroya *et al.*, 2011; Jang and Elaydi, 2003). Furthermore, this allows better use of statistical data for numerical simulations since the infection data are compiled at discrete given time intervals (Enatsu *et al.*, 2010).

The discrete models constructed by numerical methods yield a major difficulty in the calculation of numerical solutions, since they may cause Numerical Instabilities (NIs). NIs are solutions to the discrete equations that do not correspond to any solution of the original differential equations (Hildebrand, 1968;

Mickens, 1994). Discrete epidemic models contain parameters in addition to those already existing in the DEs such as the time and space step sizes. Variations in these extra parameters may cause bifurcation phenomena which show up as NIs. For example, traditional schemes like forward Euler, Runge–Kutta, and others sometimes generate oscillations and produce fictitious bifurcations, artificial chaos and false steady states (Lambert, 1973). Therefore, how to choose discrete schemes which guarantee the global dynamics of the models is a very important topic.

NSFD methods, developed by Mickens (1994; 2000; 2002; 2005), are increasingly being applied to numerical integration of differential equations (Alexander *et al.*, 2006; Dumont and Lubuma, 2005; Bruggeman *et al.*, 2007; Arenas *et al.*, 2008; Moghadas *et al.*, 2003; Anguelo and Lubuma, 2003; Dimitrov and Kojouharov, 2007; 2008; Dimitrov, 2005; Gumel, 2002; Jansen and Twizell, 2002; Chen and Clemence, 2006; Obaid *et al.*, 2013). Their use is mainly based on the fact that they are very effective in preserving certain qualitative properties of the original differential equations (Mickens, 2000; Alexander *et al.*, 2006; Sekiguchi and Ishiwata, 2010; Sekiguchi, 2010; Moghadas *et al.*, 2003). Arenas *et al.* (2008) developed a non standard numerical scheme for an SIRS seasonal epidemiological model for RSV transmission. Sekiguchi and Ishiwata (2010) derived a discretized SIRS epidemic model with time delay by

applying an NSFD scheme. With the NSFD scheme, Sekiguchi (2010) proved the permanence of an SIRS discrete epidemic model with time delays if an endemic equilibrium of each model exists. Moghadas *et al.* (2003) set forth a positivity preserving NSFD scheme for an epidemic model.

In this paper, we will show that a discrete epidemic model with vaccination constructed by NSFD unconditionally guarantee the positivity of solutions and the global dynamics of the corresponding continuous model.

The remainder of this paper is organized as follows. In the next section, we will introduce some useful definitions and a lemma for the continuous epidemic model with vaccination. In Section 3, we derive a discretized epidemic model with vaccination. We obtain a sufficient condition for the global asymptotic stability of the equilibria in Section 4. Numerical simulations are reported in Section 5.

2. Preliminaries

To prevent and control the spread of some communicable diseases, vaccination is one of the commonly used measures. Studies of epidemic models with vaccination have become an important issue in the mathematical theory of epidemiology (Ma *et al.*, 2004; Anderson and May, 1991). Vaccine-induced immunity may wane in preventive vaccines against infectious diseases, such as hepatitis B, polio and mumps, and so forth. These vaccines can only offer a period of immunity to the diseases. For example, the immunization period of the hepatitis B vaccine is about five years, and that of the rabies vaccine is about one and half years. During the immunization period, the vaccinated individuals are subjected to the natural death rate.

Let $S(t)$ be the number of susceptible individuals, $I(t)$ be the number of infective individuals, and $R(t)$ be the number of removed individuals at time t. Ma *et al.* (2004) studied the following continuous epidemic model with vaccination, which is the following system of delay differential equations:

$$
\begin{aligned}
S' &= a - dS - pS - \beta SI + pS(t-\tau)e^{-d\tau}, \\
I' &= \beta SI - (d+\alpha+\gamma)I, \\
R' &= \gamma I + pS - dR - pS(t-\tau)e^{-d\tau},
\end{aligned}
\tag{1}
$$

where a is the recruitment rate of the population, d is the natural death rate of the population, p is the vaccinating rate coefficient for a susceptible individual, β is the infection coefficient, α is the disease-induced death rate coefficient, γ is the recovery rate coefficient of the infected individuals. The immunization period is referred to as a delay τ, which means those who are vaccinated at

time $t - \tau$ would return to a susceptible individual at time t.

We adopt the following notation: $\mathbb{R}^3$ is a three-dimensional real Euclidean space with norm $|\cdot|$. For $\tau > 0$, we denote by $C = C([-\tau, 0], \mathbb{R}^3)$ the Banach space of continuous functions mapping the interval $[-\tau, 0]$ into $\mathbb{R}^3$ with the topology of uniform convergence, i.e., for $\varphi = (\varphi_1, \varphi_2, \varphi_3) \in C$, the norm of φ is defined as

$$
\| \varphi \| = \sup_{-\tau \le \theta \le 0} \{|\varphi_1(\theta)|, |\varphi_2(\theta)|, |\varphi_3(\theta)|\}.
$$

The initial conditions for the system (1) are given as

$$
\begin{aligned}
S(\theta) &= \varphi_1(\theta) \ge 0, \\
I(\theta) &= \varphi_2(\theta) \ge 0, \\
R(\theta) &= \varphi_3(\theta) \ge 0, \quad -\tau \le \theta < 0, \\
S(0) &= \varphi_1(0) > 0, \\
I(0) &= \varphi_2(0) > 0, \\
R(0) &= \varphi_3(0) > 0
\end{aligned}
\tag{2}
$$

and the solution of the system (1) is denoted by $(S(t), I(t), R(t))$.

Observe that the variable $R(t)$ does not appear in the first two equations of the system (1), and the behaviors of $R(t)$ can then be determined from the last equation in the system (1). This allows us to consider the reduced system

$$
\begin{aligned}
S' &= a - dS - pS - \beta SI + pS(t-\tau)e^{-d\tau}, \\
I' &= \beta SI - (d+\alpha+\gamma)I,
\end{aligned}
\tag{3}
$$

where the initial conditions are

$$
\begin{aligned}
S(\theta) &= \varphi_1(\theta) \ge 0, \\
I(\theta) &= \varphi_2(\theta) \ge 0, \quad -\tau \le \theta < 0, \\
S(0) &= \varphi_1(0) > 0, \\
I(0) &= \varphi_2(0) > 0.
\end{aligned}
\tag{4}
$$

Set

$$
D = \left\{ (S, I) \in \mathbb{R}^2 \,|\, S > 0, I > 0, S + I \le \frac{a}{d} \right\}.
$$

Let $\mathring{D}$ denote the interior of D. It can be verified that D is positively invariant with respect to the system (3). Ma *et al.* (2004) derived the following result.

Lemma 1. *In the system (3), $(S, I) \in D$. Set*

$$
S^0 = \frac{a}{d + p(1 - e^{-d\tau})},
$$

$$
S^* = \frac{d+\alpha+\gamma}{\beta},
$$

$$
I^* = \frac{a}{d+\alpha+\gamma}\left(1 - \frac{1}{R_0}\right)
$$

and

$$
R_0 = \frac{\beta S^0}{d+\alpha+\gamma}.
$$

Then, we have the following:

 (i) If $R_0 \leq 1$, then the system (3) has a unique disease-free equilibrium $E^0 = (S^0, 0)$ which is globally asymptotically stable in D.

 (ii) If $R_0 > 1$, then the system (3) has two equilibria, a disease-free equilibrium E^0 and an endemic equilibrium $E^* = (S^*, I^*)$. The endemic equilibrium E^* is globally asymptotically stable in $\mathring{D}$.

As is well known, it is desirable that the numerical schemes preserve the dynamics of the system (1). In particular, the numerical schemes must have at least the following properties (Mickens, 2005; Oran and Boris, 1987; Potter, 1973):

- *Positivity of solutions:* For any nonnegative initial data, the iterations must remain nonnegative.

- *Correct number and stability of equilibria*: The properties (i)–(ii) of the system (3) must hold.

Consider the nonlinear functional differential equation

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} = f(x_t), \quad t \geq 0, \tag{5}$$

where $f : C \to \mathbb{R}^n$ is continuous on compact subsets of C, and $x_t \in C$ is defined by $x_t(s) = x(t+s), s \in [-\tau, 0]$. Under this hypothesis, the solutions $x(t) = x(t, \phi)$ of Eqn. (5) are determined by the initial condition

$$x_{t_0} = \phi, \quad \text{where } \phi \in C, \ t_0 = 0. \tag{6}$$

Elements of $x(t)$ and $f(x_t)$ will be denoted by $x^i(t)$ and $f^i(x_t)$, $i = 1, 2, \ldots, n$ respectively. For any vector x, $x > 0$ will be used to denote $x^i > 0, \forall i$.

For numerical schemes, the time at integration step $n \in \mathbb{N}$ will be denoted by t_n. As this paper deals only with schemes using a fixed time step, the time step will be denoted by h, and m will be a positive integer satisfying $\tau = mh$. The numerical approximation of the solution vector $x(t_n)$ will be denoted by x_n. According to Bruggeman *et al.* (2007) as well as Moghadas and Gumel (2003), we give the following definition.

Definition 1. (*Unconditional positivity of a system*) A system (5) is called unconditionally positive if $f(x_t)$ is such that $x(t) > 0$ for all $t > 0$, with $\phi \in C$ satisfying

$$\phi(s) \geq 0, \quad s \in [-\tau, 0], \quad \phi(0) > 0. \tag{7}$$

Definition 2. (*Unconditional positivity of a numerical scheme*) A numerical scheme ψ for the system (5) is called unconditionally positive if $x_n > 0, n = 1, 2, \ldots$ for any given $x_k = \phi(-kh) > 0, k = 0, 1, 2, \ldots, m$, with $\phi \in C$ satisfying (7).

Definition 3. (*Unconditional stability of a numerical scheme*) A numerical scheme Ψ is called unconditionally stable if it is stable irrespective of the value of the step size used to simulate it.

Applying Mickens' nonstandard discretization to the continuous model (1), Ding *et al.* (2013), proposed the following discretized epidemic model with vaccination

$$
\begin{aligned}
\frac{S_{n+1} - S_n}{h} &= a - dS_n - pS_{n+1} - \beta S_{n+1} I_n \\
&\quad + pS_{n-m} e^{-d\tau}, \\
\frac{I_{n+1} - I_n}{h} &= \beta S_{n+1} I_n - (d + \alpha + \gamma) I_n, \\
\frac{R_{n+1} - R_n}{h} &= \gamma I_n + pS_{n+1} - dR_n - pS_{n-m} e^{-d\tau}.
\end{aligned}
\tag{8}
$$

There, they show that, when

$$h < h_0 = \frac{1}{d + \alpha + \gamma},$$

the difference equation system (8) can keep the global dynamics of the continuous system (1). However, the results in Ding *et al.* (2013) are limited by the step size. In this paper, this restriction is removed by constructing more proper nonstandard scheme.

3. Unconditionally positive discretized model

In this section, denote by S_n the susceptible class, by I_n the infective class and by R_n the recovered class at the n-th step, respectively. Applying Mickens' nonstandard discretization to the continuous model (1), we derive the following discretized epidemic model with vaccination, which is a system of difference-delay equations:

$$
\begin{aligned}
\frac{S_{n+1} - S_n}{h} &= a - dS_{n+1} - pS_{n+1} - \beta S_{n+1} I_n \\
&\quad + pS_{n-m} e^{-d\tau}, \\
\frac{I_{n+1} - I_n}{h} &= \beta S_{n+1} I_n - (d + \alpha + \gamma) I_{n+1}, \\
\frac{R_{n+1} - R_n}{h} &= \gamma I_{n+1} + pS_{n+1} - dR_{n+1} \\
&\quad - pS_{n-m} e^{-d\tau},
\end{aligned}
\tag{9}
$$

where $n = 0, 1, \ldots$. The initial conditions of the system (9) are given by

$$
\begin{aligned}
S(-k) &= \varphi_1(-kh) \geq 0, \\
S(0) &> 0, \quad I(0) > 0, \\
R(0) &> 0, \quad k = 1, 2, \ldots, m.
\end{aligned}
\tag{10}
$$

Observe that the variable R_n does not appear in the first two equations of the system (9), and the behaviors of

R_n can then be determined from the last equation in the system (9). This allows us to consider the reduced system

$$\frac{S_{n+1} - S_n}{h} = a - (d+p)S_{n+1} - \beta S_{n+1}I_n$$
$$+ pS_{n-m}e^{-d\tau}, \qquad (11)$$
$$\frac{I_{n+1} - I_n}{h} = \beta S_{n+1}I_n - (d+\alpha+\gamma)I_{n+1},$$

where the initial conditions are

$$S(-k) \geq 0, \quad S(0) > 0, \quad I(0) > 0,$$
$$k = 1, 2, \ldots, m. \qquad (12)$$

After some simple manipulations, the system (11) is transformed into the following form:

$$S_{n+1} = \frac{ha + S_n + hpS_{n-m}e^{-d\tau}}{1 + (d+p)h + \beta h I_n},$$
$$I_{n+1} = \frac{\beta h S_{n+1}I_n + I_n}{[1 + h(d+\alpha+\gamma)]}. \qquad (13)$$

It is easy to get the following lemma.

Lemma 2. *The system* (11) *with the initial condition* (12) *is unconditionally positive.*

Set $N_n = S_n + I_n + R_n$. Then

$$N_{n+1} = \frac{ah + N_n - \alpha h I_{n+1}}{1 + dh} \leq \frac{ah + N_n}{1 + dh}. \qquad (14)$$

Therefore,

$$N_n \leq \max\left\{N_0, \frac{a}{d}\right\}$$

and

$$\limsup_{n\to\infty} N_n \leq \frac{a}{d}.$$

Theorem 1. *For any solution* (S_n, I_n, R_n) *of the system* (9), *the total population* N_n *satisfies*

$$\limsup_{n\to\infty} N_n \leq \frac{a}{d}. \qquad (15)$$

Define

$$\Gamma = \left\{(S_n, I_n) \in \mathbb{R}^2 | 0 < S_n, \right.$$
$$\left. 0 < I_n, 0 < S_n + I_n \leq \frac{a}{d}\right\},$$

and let $\mathring{\Gamma}$ denote the interior of Γ. It can be verified that Γ is positively invariant with respect to the system (11). Our results will be stated for the system (11) in Γ.

4. Global asymptotic stability of the equilibria

In this section, we will show that the system (11) is unconditionally stable.

An equilibrium of the system (11) satisfies

$$0 = a - (d+p)S - \beta SI + pSe^{-d\tau},$$
$$0 = \beta SI - (d+\alpha+\gamma)I. \qquad (16)$$

It is easy to verify that the equilibria of the discretized system (9) are the equilibria of the system (1). Furthermore, the equilibria of the system (11) are the same as (3), namely, E^0 and E^*.

In this section, firstly, we obtain a sufficient condition for global asymptotic stability of the disease-free equilibrium E^0 in the system (11) by a Lyapunov function.

Theorem 2. *In the system* (11), *we have the following:*

(*i*) *If* $R_0 \leq 1$, *the disease-free equilibrium* $E^0 = (S^0, 0)$ *of the system (11) is globally asymptotically stable in* Γ.

(*ii*) *If* $R_0 > 1$, E^0 *is unstable.*

Proof. Part (i). It is convenient to translate the disease-free equilibrium E^0 to the origin by setting $X_n = S_n - S^0$. By Eqn.(16), we know S^0 satisfies $a - (d+p)S^0 + pS^0e^{-d\tau} = 0$. Then Eqn.(11) becomes

$$\frac{X_{n+1} - X_n}{h} = -dX_{n+1} - pX_{n+1} - \beta X_{n+1}I_n$$
$$- \beta S^0 I_n + pX_{n-m}e^{-d\tau},$$
$$\frac{I_{n+1} - I_n}{h} = \beta X_{n+1}I_n + \beta S^0 I_n \qquad (17)$$
$$- (d+\alpha+\gamma)I_{n+1}.$$

Define

$$L|^1_{(11)n} = \frac{X_n^2}{2} + [1 + (d+\alpha+\gamma)h]S^0 I_n. \qquad (18)$$

We obtain

$$\Delta L|^1_{(11)n}$$
$$= \frac{1}{2}(X_{n+1}^2 - X_n^2) + [h\beta S^0 X_{n+1}I_n$$
$$+ h\beta S^{0^2}I_n - (d+\alpha+\gamma)hS^0 I_n]$$
$$= \frac{1}{2}\left\{X_{n+1}^2 - \left[X_{n+1} + ((d+p)hX_{n+1}\right.\right.$$
$$\left.\left. + \beta h X_{n+1}I_n + \beta h S^0 I_n - phX_{n-m}e^{-d\tau})\right]^2\right\}$$

$$+ S^0(\beta h X_{n+1} I_n + \beta h S^0 I_n - (d + \alpha + \gamma)h I_n)$$
$$\leq - X_{n+1}((d + p)h X_{n+1} + \beta h X_{n+1} I_n$$
$$+ \beta h S^0 I_n - ph X_{n-m} e^{-d\tau}) + S^0(\beta h X_{n+1} I_n$$
$$+ \beta h S^0 I_n - (d + \alpha + \gamma)h I_n)$$
$$= - (d + p)h X_{n+1}^2 - \beta h X_{n+1}^2 I_n$$
$$+ ph X_{n+1} X_{n-m} e^{-d\tau} + \beta h S^{0^2} I_n$$
$$- (d + \gamma + \alpha)h S^0 I_n. \tag{19}$$

Then

$$\Delta L|_{(11)n}^1 \leq - (d + p)h X_{n+1}^2 - \beta h X_{n+1}^2 I_n$$
$$+ \frac{ph e^{-d\tau}}{2} X_{n+1}^2 + \frac{ph e^{-d\tau}}{2} X_{n-m}^2 \tag{20}$$
$$+ \beta h S^{0^2} I_n - (d + \alpha + \gamma)h S^0 I_n.$$

Define

$$L|_{(11)n}^2 = \frac{hp e^{-d\tau}}{2} \sum_{k=n-m}^{n} X_k^2. \tag{21}$$

Similarly, we obtain

$$\Delta L|_{(11)n}^2 = \frac{hp e^{-d\tau}}{2} \left(\sum_{k=n+1-m}^{n+1} X_k^2 - \sum_{k=n-m}^{n} X_k^2 \right)$$
$$= \frac{hp e^{-d\tau}}{2} (X_{n+1}^2 - X_{n-m}^2). \tag{22}$$

Set $L|_{(11)n} = L|_{(11)n}^1 + L|_{(11)n}^2$. Then we get

$$\Delta L|_{(11)n}$$
$$= \Delta L|_{(11)n}^1 + \Delta L|_{(11)n}^2$$
$$\leq -[d + p(1 - e^{-d\tau})]h X_{n+1}^2 - \beta h X_{n+1}^2 I_n$$
$$+ \beta h S^{0^2} I_n - (d + \alpha + \gamma)h S^0 I_n$$
$$\leq -[d + p(1 - e^{-d\tau})]h X_{n+1}^2$$
$$- S^0 h(d + \alpha + \gamma - \beta S^0)I_n \tag{23}$$
$$\leq -[d + p(1 - e^{-d\tau})]h X_{n+1}^2$$
$$- S^0 h(d + \alpha + \gamma)(1 - R_0)I_n.$$

Obviously, when $R_0 \leq 1$, $\Delta L|_{(11)n} \leq 0$; when $R_0 < 1$, $\Delta L|_{(11)n} = 0 \Leftrightarrow X_{n+1} = X_{n-m} = 0$, $I_n = 0$; when $R_0 = 1$, $\Delta L|_{(11)n} = 0 \Leftrightarrow X_{n+1} = X_{n-m} = 0$. Substituting $R_0 = 1$ and $X_{n+1} = X_{n-m} = 0$ into the system (17), we have $I_n = I_{n+1}$. That is, the only compact invariant set where $\{\triangle L|_{(11)n} = 0\}$ is the singleton $\{X_n = 0, I_n = 0\}$. By Lyapunov functions and stability (Stuart and Humphries, 1996), we have that E^0 is globally asymptotically stable in Γ, if $R_0 \leq 1$.

Part (ii). If $R_0 > 1$, the linearization of Eqn. (17) at E^0 is

$$\frac{X_{n+1} - X_n}{h} = -d X_{n+1} - p X_{n+1} - \beta S^0 I_n$$
$$+ p X_{n-m} e^{-d\tau}, \tag{24}$$
$$\frac{I_{n+1} - I_n}{h} = \beta S^0 I_n - (d + \alpha + \gamma)I_{n+1}.$$

Then the characteristic equation takes the form

$$\begin{vmatrix} \tilde{\vartheta}_1 & \beta h S^0 \lambda^m \\ 0 & \tilde{\vartheta}_2 \end{vmatrix} = 0, \tag{25}$$

where

$$\tilde{\vartheta}_1 = [1 + (d + p)h]\lambda^{m+1} - \lambda^m - ph e^{-d\tau},$$

$$\tilde{\vartheta}_2 = \lambda^{m+1} - \frac{1 + (\alpha + d + \gamma)R_0 h}{1 + (\alpha + d + \gamma)h} \lambda^m.$$

It is easy to demonstrate that

$$\lambda_0 = \frac{1 + (\alpha + d + \gamma)R_0 h}{1 + (\alpha + d + \gamma)h}$$

is one of the roots of Eqn. (25). Clearly, if $R_0 > 1$, $\lambda_0 > 1$. Therefore, if $R_0 > 1$, the disease-free equilibrium E^0 of the system (11) is unstable. ∎

Next, we will show the global stability for the endemic equilibrium E^* of the system (11) by a Lyapunov function.

Theorem 3. *If $R_0 > 1$, the endemic equilibrium $E^* = (S^*, I^*)$ of the system (11) is globally asymptotically stable in $\hat{\Gamma}$.*

Proof. Set

$$V|_{(11)n} = V|_{(11)n}^1 + V|_{(11)n}^2, \tag{26}$$

where

$$V|_{(11)n}^1 = S_n - S^* - S^* \ln \frac{S_n}{S^*}$$
$$+ [1 + (\alpha + d + \gamma)h]$$
$$\times \left(I_n - I^* - I^* \ln \frac{I_n}{I^*} \right), \tag{27}$$
$$V|_{(11)n}^2 = ph e^{-d\tau} \sum_{k=-m}^{0} \left(S_{n-k} - S^* \right)$$
$$- S^* \ln \frac{S_{n-k}}{S^*} \right).$$

Set $\psi(x) = 1 + \ln x - x$ for all $x > 0$. We know that

$$\psi(x) \leq 0 \quad \text{for} \quad x > 0, \tag{28}$$

with the equality holding if and only if $x = 1$. Then we get

$$
\begin{aligned}
\Delta V&|_{(11)n}^1 \\
&= \left[S_{n+1} - S_n + S^* \ln \frac{S_n}{S_{n+1}} \right] \\
&\quad + [1 + (\alpha + d + \gamma)h]\left[I_{n+1} - I_n + I^* \ln \frac{I_n}{I_{n+1}} \right] \\
&= \left[S_{n+1} - S_n + S^* \left(\psi\left(\frac{S_n}{S_{n+1}}\right) - 1 + \frac{S_n}{S_{n+1}} \right) \right] \\
&\quad + [1 + (\alpha + d + \gamma)h]\Big[I_{n+1} - I_n \\
&\qquad + I^*\left(\psi\left(\frac{I_n}{I_{n+1}}\right) - 1 + \frac{I_n}{I_{n+1}} \right) \Big] \\
&\leq \left(1 - \frac{S^*}{S_{n+1}} \right)(S_{n+1} - S_n) \\
&\quad + [1 + (\alpha + d + \gamma)h]\left[\left(1 - \frac{I^*}{I_{n+1}} \right)(I_{n+1} - I_n) \right].
\end{aligned}
$$

Together with Eqns. (11) and (16), we produce

$$
\begin{aligned}
\Delta V&|_{(11)n}^1 \\
&\leq \left(1 - \frac{S^*}{S_{n+1}} \right)[(d+p)hS^* + \beta h S^* I^* - ph S^* e^{-d\tau} \\
&\quad - (d+p)hS_{n+1} - \beta h S_{n+1} I_n + ph S_{n-m} e^{-d\tau}] \\
&\quad + \left(1 - \frac{I^*}{I_{n+1}} \right)[\beta h S_{n+1} I_n - (d + \gamma + \alpha)h I_n].
\end{aligned} \tag{29}
$$

Similarly, we have

$$
\begin{aligned}
\Delta V&|_{(11)n}^2 \\
&= phe^{-d\tau} \sum_{k=-m}^{0} \left(S_{n+1-k} - S_{n-k} + S^* \ln \frac{S_{n-k}}{S_{n+1-k}} \right) \\
&= phe^{-d\tau}\left(S_{n+1} - S_{n-m} + S^* \ln \frac{S_{n-m}}{S_{n+1}} \right) \\
&= phe^{-d\tau}\Big[S_{n+1} - S_{n-m} \\
&\quad + S^*\left(\psi\left(\frac{S_{n-m}}{S_{n+1}}\right) - 1 + \frac{S_{n-m}}{S_{n+1}} \right) \Big] \\
&\leq phe^{-d\tau}\left(S_{n+1} - S_{n-m} - S^* \frac{S_{n+1} - S_{n-m}}{S_{n+1}} \right).
\end{aligned} \tag{30}
$$

By Eqns. (16), (29) and (30), and using the fact that

$$
\frac{S^*}{S_{n+1}} + \frac{S_{n+1}}{S^*} \geq 2, \tag{31}
$$

with the equality holding if and only if $S^* = S_{n+1}$, we

obtain

$$
\begin{aligned}
\Delta V&|_{(11)n} \\
&= \Delta V|_{(11)n}^1 + \Delta V|_{(11)n}^2 \\
&\leq hS^*[d + p(1 - e^{-d\tau})]\left(2 - \frac{S^*}{S_{n+1}} - \frac{S_{n+1}}{S^*} \right) \\
&\quad + \left(\beta h S^* I^* - \beta h S_{n+1} I_n - \beta h \frac{S^{*2} I^*}{S_{n+1}} + \beta h S^* I_n \right) \\
&\quad + \left(1 - \frac{I^*}{I_{n+1}} \right)[\beta h S_{n+1} I_n - (d + \alpha + \gamma)h I_n] \\
&\leq \beta h \Big[S^* I^* - S_{n+1} I_n - \frac{S^{*2} I^*}{S_{n+1}} + S^* I_n \\
&\quad + \left(1 - \frac{I^*}{I_{n+1}} \right)(S_{n+1} I_n - S^* I_n) \Big] \\
&= \beta h \left(S^* I^* - \frac{S^{*2} I^*}{S_{n+1}} - \frac{S_{n+1} I_n}{I_{n+1}} I^* + \frac{S^* I^* I_n}{I_{n+1}} \right) \\
&= \beta h I^* \left(\frac{S^*}{S_{n+1}} - \frac{I_n}{I_{n+1}} \right)(S_{n+1} - S^*) \\
&= I^*(1 + \beta h S^*)\left(\frac{S^*}{S_{n+1}} - \frac{I_n}{I_{n+1}} \right)\left(\frac{I_{n+1}}{I_n} - 1 \right).
\end{aligned}
$$

From Eqn. (11), it is easy to see that

$$
I_{n+1} - I_n = \frac{1}{1 + \beta S^* h}(\beta h S_{n+1} I_n - \beta S^* h I_n). \tag{32}
$$

By Eqn. (32), we get

$$
\begin{aligned}
(1 + \beta S^* h)&\left(\frac{I_{n+1}}{I_n} - 1 \right) > -\beta h S^*, \\
\frac{S^*}{S_{n+1}} &= \frac{\beta h S^*}{\left(\frac{I_{n+1}}{I_n} - 1 \right)(1 + \beta S^* h) + \beta h S^*}.
\end{aligned} \tag{33}
$$

Set $x = I_{n+1}/I_n$. Then, by Eqn. (33), we have

$$
\begin{aligned}
\Delta V&|_{(11)n} \\
&\leq I^*(1 + \beta h S^*) \\
&\quad \times \left[\frac{\beta h S^*}{(x-1)(1 + \beta S^* h) + \beta h S^*} - \frac{1}{x} \right](x - 1) \\
&= -\frac{I^*(1 + \beta h S^*)(x - 1)^2}{x[(x-1)(1 + \beta S^* h) + \beta h S^*]} \leq 0
\end{aligned} \tag{34}
$$

According to Eqns. (34), (28) and (31), we know $\Delta V|_{(11)n} = 0 \Leftrightarrow S_{n+1} = S_n = S_{n-m} = S^*$ and $I_{n+1} = I_n$. Substituting this into Eqn. (11), we get the only compact invariant set, where $\{\Delta V|_{(11)n} = 0\}$ is the singleton $\{E^*\}$. By LaSalle's invariance principle (Sundarapandian, 2003), E^* is globally asymptotically stable in $\overset{\circ}{\Gamma}$. ∎

Remark 1. From Theorems 2 and 3, it cannot happen that S_n is exponential decay. Thus, by the third equation of (9), it should be easy to show that $R_n > 0$ with the initial

condition (12). Together with Lemma 2, we know our results in this paper can be translated straightforwardly to the system (9).

5. Numerical simulation

For the system (9), Theorem 2 implies that the disease decays if $R_0 \leq 1$, and Theorem 3 implies that the disease persists if $R_0 > 1$. In order to illustrate the validity of our results, we consider the following two epidemic models with vaccination. Firstly,

$$
\begin{aligned}
S' &= 14 - 0.3S - 0.6S - 0.04SI \\
&\quad + 0.6S(t-10)e^{-0.3 \times 10}, \\
I' &= 0.04SI - (0.3 + 0.009 + 0.9)I, \quad (35) \\
R' &= 0.9I + 0.6S - 0.3R \\
&\quad - 0.6S(t-10)e^{-0.3 \times 10}.
\end{aligned}
$$

After some simple manipulations, we get $R_0 = 0.5276 < 1$ and Eqn. (35) has a unique equilibrium $E^0 = (S^0, I^0, R^0) = (15.95, 0, 30.72)$. By Lemma 1(i), E^0 is globally asymptotically stable.

Then

$$
\begin{aligned}
S' &= 10 - 0.4S - 0.5S - 0.5SI \\
&\quad + 0.5S(t-10)e^{-0.4 \times 10}, \\
I' &= 0.5SI - (0.4 + 0.4 + 0.6)I, \quad (36) \\
R' &= 0.6I + 0.5S - 0.4R \\
&\quad - 0.5S(t-10)e^{-0.4 \times 10}.
\end{aligned}
$$

It is easy to see that $R_0 = 3.9955 > 1$ and Eqn. (35) has two equilibria,

$$E^0 = (S^0, I^0, R^0) = (11.1874, 0, 13.8126),$$

$$E^* = (S^*, I^*, R^*) = (2.8, 5.3551, 11.4897).$$

By Lemma 1(ii), E^* is globally asymptotically stable.

In Figs. 1–6, the numbers of susceptible, infective and recovered individuals (on the vertical axis) are plotted versus the time steps n (on the horizontal axis). Figure 1 shows that the solutions of the system (35) under the numerical schemes (9) converge to the disease-free equilibrium when $h = 0.1, 1, 11$. We can see that the disease free equilibrium E^0 of the system (35) is globally asymptotically stable. On the other hand, Fig. 4 shows that the solutions of the system (36) under the method (9) converge to the endemic equilibrium when $h = 0.1, 1, 11$, and indicates that E^* of the system (36) is globally asymptotically stable.

In the following, we give some comparison of the numerical scheme (8) with the scheme (9). Firstly, we use the NSFD method (8) to calculate numerical solutions for the underlying system (1). With the system (35), moreover, when the step size reaches a certain degree,

the numerical solution exhibits oscillations of period two (see Fig. 2(a)), and even completely loses stability (see Fig. 3(a)). With the growth of the step size, the situation of the system (36) is analogous to that of the system (35) (see Figs. 5(a) and 6(a)). Then, we also use the NSFD method (9) with the systems (35) and (36). It is shown that the stability of the disease free equilibrium E^0 and endemic equilibrium E^* have not been influenced even though the step size grows (see Figs.1(c), 2(b), 3(b) and 4(c), 5(b), 6(b)).

In addition, we make a comparison between the 4-th order Runge–Kutta method and the numerical scheme (9) for the system (35). It can be seen that the 4-th order Runge–Kutta method did not converge (see Fig.7(b)) or in some cases took unreal negative values (see Fig.7(a)) for the infective population, while the numerical scheme (9) converges to the correct endemic point and only has positive solutions with the same time step size.

In order to further illustrate the advantages of the proposed numerical scheme (9), we take the following parameter values: $a = 0.002$, $p = 1$, $d = 0.002$, $b = 500$, $\alpha = 20$ and $\gamma = 1$. In Fig. 8 for $R_0 > 1$, it can be observed that the numerical scheme (9) converges correctly to the endemic equilibrium point and only produces positive values for all times t. However, the routines of the Matlab software program with their default error tolerance did not converge.

6. Conclusion and discussion

In this paper, we derived a discretized epidemic model with vaccination by the NSFD method. For any $h > 0$, we showed the positivity of numerical solutions and obtained sufficient conditions for global behaviors of the equilibria by a Lyapunov function. Our results demonstrate that the dynamic behaviors of a discretized system is in keeping with the continuous one.
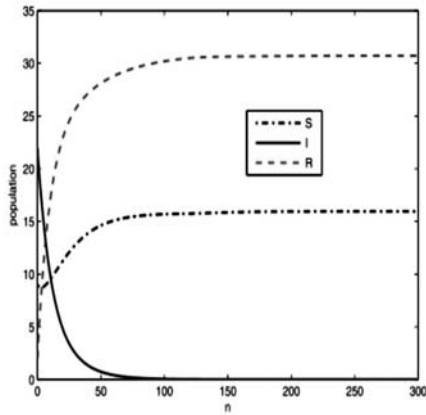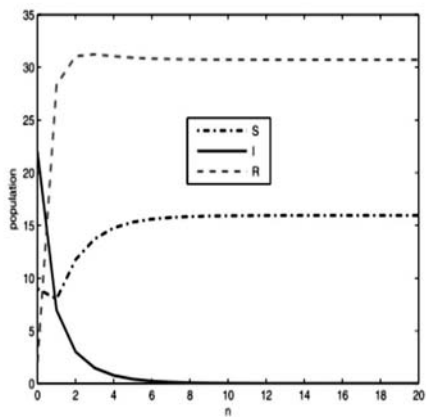
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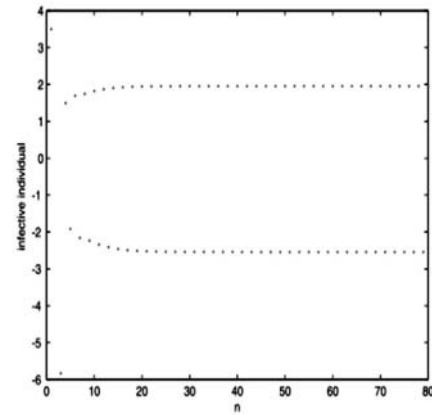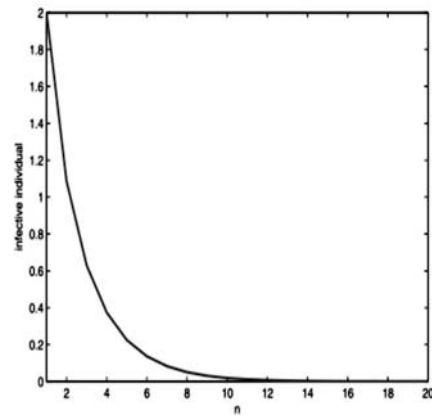
(a)

(b)

(c)

Fig. 1. Solution of the system (35) under the numerical scheme (9): $h = 0.1$ (a), $h = 1$ (b) and $h = 11$ (c).

(a)

(b)

Fig. 2. Solution of the system (35) with $h = 3.67$ on the (I, n) plane: under the numerical scheme (8) (a), under the numerical scheme (9) (b).

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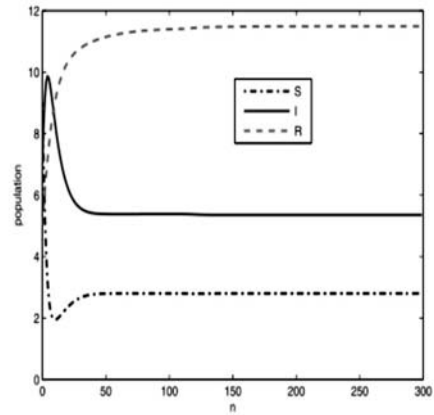
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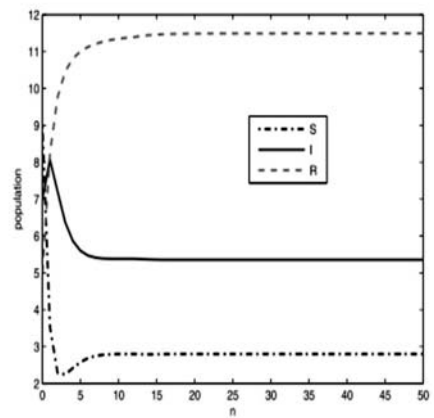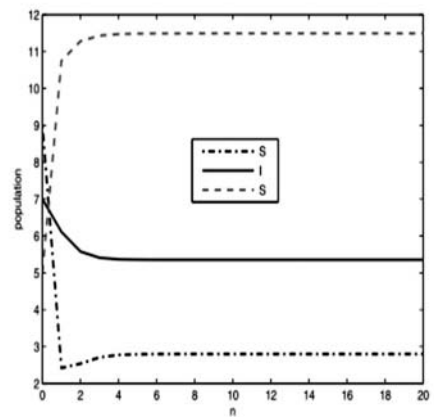
(a)

(a)

(b)

(b)

(c)

Fig. 3. Solution of the system (35) with $h = 5.5$ on the (I, n) plane: under the numerical scheme (8) (a), under the numerical scheme (9) (b).

Fig. 4. Solution of the system (36) under the numerical scheme (9): $h = 0.1$ (a), $h = 1$ (b), $h = 11$ (c).

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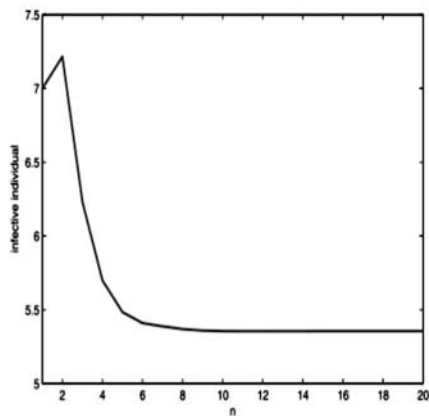
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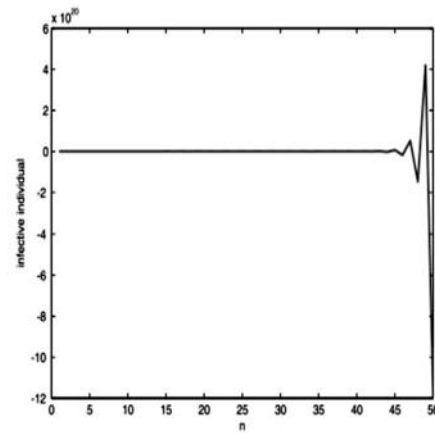
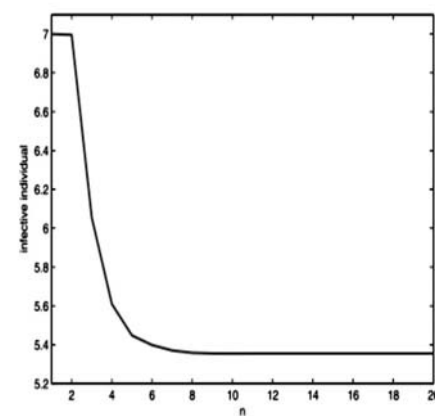Enszer, J.A. and Stadtherr, M.A. (2009). Verified solution

(a)

(b)

Fig. 5. Solution of the system (36) with $h = 2.2$ on the (I, n) plane: under the numerical scheme (8) (a), under the numerical scheme (9) (b).

(a)

(b)

Fig. 6. Solution of the system (36) with $h = 2.75$ on the (I, n) plane: under the numerical scheme (8) (a), under the numerical scheme (9) (b).

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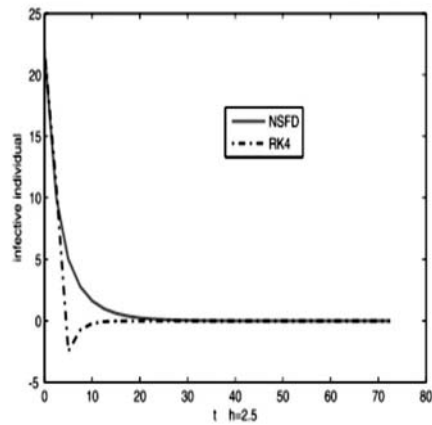
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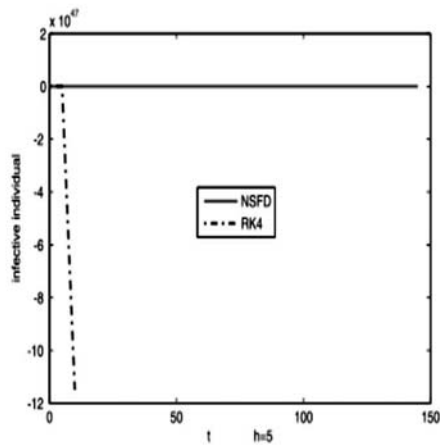
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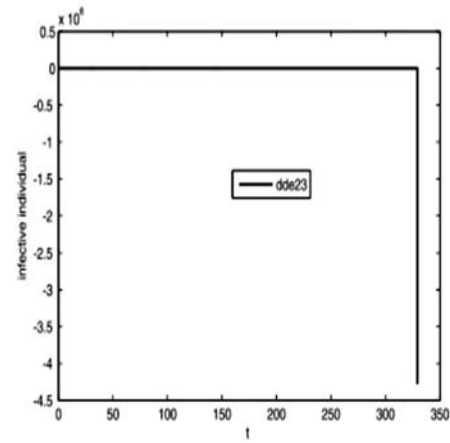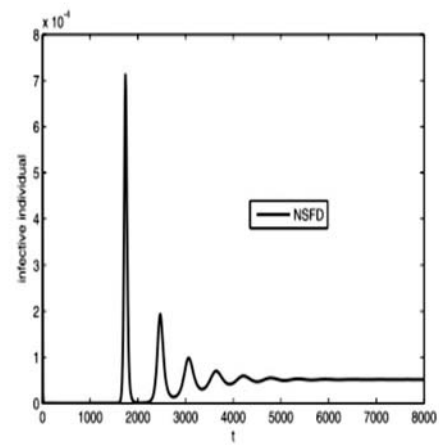
(a)

(b)

Fig. 7. Profile of infective individuals $I(t)$ using the 4-th order Runge–Kutta method (RK4) and the nonstandard numerical scheme (NSFD) (9) with the system (35).

(a)

(b)

Fig. 8. Profile of an infected subpopulation $I(t)$ with the parameter values $a = 0.002$, $p = 1$, $d = 0.002$, $b = 500$, $\alpha = 20$ and $\gamma = 1$ ($R_0 > 1$): using the Matlab mathematical package (a), under the numerical scheme (9) (b).

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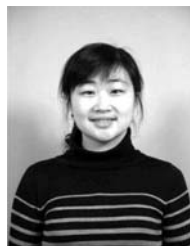
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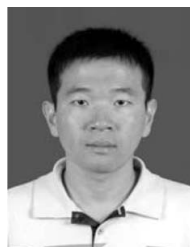
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