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POSITIVITY AND LINEARIZATION OF A CLASS OF NONLINEAR CONTINUOUS–TIME SYSTEMS BY STATE FEEDBACKS

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The positivity and linearization of a class of nonlinear continuous-time system by nonlinear state feedbacks are addressed. Necessary and sufficient conditions for the positivity of the class of nonlinear systems are established. A method for linearization of nonlinear systems by nonlinear state feedbacks is presented. It is shown that by a suitable choice of the state feedback it is possible to obtain an asymptotically stable and controllable linear system, and if the closed-loop system is positive then it is unstable.

Keywords: positive, nonlinear, system, linearization, state feedback.

1. Introduction

In positive systems inputs, state variables and outputs take only nonnegative values. Examples of positive systems are industrial processes involving chemical reactors, heat exchangers and distillation columns, storage systems, compartmental systems, water and atmospheric pollution models. A variety of models having positive linear behavior can be found in engineering, management science, economics, social sciences, biology and medicine, etc. Positive linear systems are defined on cones and not on linear spaces. Therefore, the theory of positive systems is more complicated and less advanced.

An overview of the state of the art in positive systems theory is given by Farina and Rinaldi (2000) as well as Kaczorek (2002), who also addressed positive linear systems consisting of n subsystems with different fractional orders (Kaczorek, 2011; 2012) along with minimum energy control of positive discrete-time and continuous-time linear systems (Kaczorek, 2014a; 2013; 2014b; 2014c). The theory of the geometrical approach to the analysis of nonlinear systems based on the Lie algebra was given by Brockett (1976) and Isidori (1989). The problem of linearization of nonlinear systems by nonlinear state feedbacks was investigated by Aguilar *et al.* (1995), Charlet *et al.* (1991), Daizhan *et al.* (1985), Fang and Kelkar (2003), Jakubczyk (2001), Jakubczyk and Respondek (1980), Isidori (1989), Malesza (2008), Marino and Tomei (1995), Melhem *et al.* (2006), Taylor and Antoniotti (1993), as well as Wei-Bing and Dang-Nan (1992).

In this paper the positivity and linearization of a class of nonlinear continuous-time systems by nonlinear state feedbacks will be addressed. The paper is organized as follows. In Section 2, necessary and sufficient conditions for the positivity of a class of nonlinear systems are established. Linearization of the nonlinear system by a nonlinear state feedback is addressed in Section 3. An example illustrating the discussion is given in Section 4. Concluding remarks are presented in Section 5.

The following notation will be used: $\mathbb{R}$, the set of real numbers; $\mathbb{R}^{n\times m}$, the set of $n \times m$ real matrices and $\mathbb{R}^n = \mathbb{R}^{n\times 1}$; $\mathbb{R}_+^{n\times m}$, the set of $n \times m$ matrices with nonnegative entries and $\mathbb{R}_+^n = \mathbb{R}_+^{n\times 1}$; M_n, the set of $n \times n$ Metzler matrices (with nonnegative off-diagonal entries); I_n, the $n \times n$ identity matrix.

2. Positivity of nonlinear systems

Consider the nonlinear system

$$\dot{x} = Ax + f(x) + Bu, \tag{1}$$

where

$$x = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}, \quad A = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ 0 & 0 & 0 & \dots & 0 \end{bmatrix}, \qquad (2)$$

$$f(x) = \begin{bmatrix} f_1(x_1) \\ f_2(x_1, x_2) \\ \vdots \\ f_n(x_1, \dots, x_n) \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix},$$

$x = x(t) \in \mathbb{R}^n$, $u = u(t) \in \mathbb{R}$ are the state vector and the input vector, respectively.

It is assumed that the functions $f_k(x_1, \dots, x_k)$, $k = 1, 2, \dots, n$, are continuously differentiable for all their arguments.

Definition 1. The nonlinear system (1) is called (internally) *positive* if $x(t) \in \mathbb{R}^n_+$ for all $x(0) \in \mathbb{R}^n_+$, $t \geq 0$ and every $u(t) \in \mathbb{R}_+$, $t \geq 0$.

Theorem 1. *The nonlinear system (1) is positive if and only if*

$$\left.\begin{array}{l} f_k(\bar{x}) \in \mathbb{R}_+ \text{ for} \\ \quad \bar{x} = [x_1, \dots, x_{j-1}, 0, x_{j+1}, \dots, x_k(t)]^T \in \mathbb{R}_+, \\ \quad j = 1, 2, \dots, k \text{ and } u(t) \in \mathbb{R}_+,\ t \geq 0. \end{array}\right\} \qquad (3)$$

Proof. For given $f(x)$, the solution of (1) has the form

$$x(t) = e^{At}x(0) + \int_0^t e^{A(t-\tau)}[f(x(\tau)) + Bu(\tau)]\,\mathrm{d}\tau. \qquad (4)$$

The linear system obtained from (1) for $f_k(x_1, \dots, x_k) = 0$, $k = 1, 2, \dots, n$, is positive since the matrix A is a Meltzer matrix, $B \in \mathbb{R}^n_+$.

Using the well-known Picard method, the k-approximation of the solution of (1) can be found from the formula

$$\begin{aligned} x_{k+1}(t) &= e^{At}x(0) \\ &\quad + \int_0^t e^{A(t-\tau)}[f(x_k(\tau)) + Bu(\tau)]\,\mathrm{d}\tau \end{aligned} \qquad (5)$$

for $k = 1, 2, \dots$.

The Lipschitz conditions for (1) are satisfied since, by assumption, the functions $f_k(x_1, \dots, x_k)$, $k = 1, 2, \dots, n$, are continuously differentiable. Using the Picard method, it is easy to show that Eqn. (1) has nonnegative solution $x(t) \in \mathbb{R}^n_+$, $t \geq 0$ if and only if the conditions (3) are satisfied. ■

The proof can be also accomplished using the method presented by Malesza and Respondek (2007).

3. Linearization by state feedbacks

For the nonlinear system (1), we introduce the following new state variables (the components of the new state vector $z = [\, z_1 \ \dots \ z_n \,]^T$):

$$\begin{aligned} z_1 &= x_1, \\ z_2 &= x_2 + f_1(x_1), \\ z_3 &= x_3 + f_2(x_1, x_2) + \frac{\partial f_1}{\partial x_1}[x_2 + f_1(x_1)] \\ &= x_3 + \bar{f}_2(x_1, x_2), \\ z_4 &= x_4 + f_3(x_1, x_2, x_3) + \frac{\partial \bar{f}_2}{\partial x_1}[x_2 + f_1(x_1)] \\ &\quad + \frac{\partial \bar{f}_2}{\partial x_2}[x_3 + f_2(x_1, x_2)] \\ &= x_4 + \bar{f}_3(x_1, x_2, x_3), \\ &\ \ \vdots \\ z_n &= x_n + \bar{f}_{n-1}(x_1, \dots, x_{n-1}). \end{aligned} \qquad (6)$$

The relations (6) can be written shortly as $z = \phi(x)$.

From (6), we have

$$\begin{aligned} x_1 &= z_1, \\ x_2 &= z_2 - f_1(z_1), \\ x_3 &= z_3 - \bar{f}_2(z_1, z_2), \\ &\ \ \vdots \\ x_n &= z_n - \bar{f}_{n-1}(z_1, \dots, z_{n-1}). \end{aligned} \qquad (7)$$

This can be briefly expressed as $x = \phi^{-1}(z)$.

The nonlinear system (1) in the new state variables (6) has the form

$$\begin{aligned} \dot{z}_1 &= \dot{x}_1 = x_2 + f_1(x_1) = z_2, \\ \dot{z}_2 &= \dot{x}_2 + \frac{\partial f_1}{\partial x_1}\dot{x}_1 = x_3 + f_2(x_1, x_2) \\ &\quad + \frac{\partial f_1}{\partial x_1}[x_2 + f_1(x_1)] = z_3, \\ &\ \ \vdots \\ \dot{z}_{n-1} &= x_n + f_{n-1}(x_1, \dots, x_{n-1}) \\ &\quad + \frac{\partial \bar{f}_{n-2}}{\partial x_1}[x_2 + f_1(x_1)] + \dots + \frac{\partial \bar{f}_{n-2}}{\partial x_{n-2}} \\ &\quad \times [x_{n-1} f_{n-2}(x_1, \dots, x_{n-2})] = z_n, \\ \dot{z}_n &= f_n(x_1, \dots, x_n) + u + \frac{\partial \bar{f}_{n-1}}{\partial x_1}[x_2 + f_1(x_1)] \\ &\quad + \dots + \frac{\partial \bar{f}_{n-1}}{\partial x_{n-1}} \\ &\quad \times [x_n + f_{n-1}(x_1, \dots, x_{n-1})]\Big|_{x=\phi^{-1}(z)} \\ &= -a_0 z_1 - a_1 z_2 - \dots - a_{n-1} z_n + v, \end{aligned} \qquad (8)$$

where

$$\begin{aligned} v &= u + g(x), \\ g(x) &= \sum_{i=0}^{n-1} a_i z_{i+1}\Big|_{z=\phi(x)} + f_n(x_1,\dots,x_n) \\ &\quad + \frac{\partial \bar{f}_{n-1}}{\partial x_1}[x_2 + f_1(x_1)] \\ &\quad + \dots + \frac{\partial \bar{f}_{n-1}}{\partial x_{n-1}}[x_n + f_{n-1}(x_1,\dots,x_{n-1})]. \end{aligned} \tag{9}$$

Equations (8) can be written in the form

$$\dot{z} = Az + Bv, \quad z(0) = \phi[x(0)] \in \mathbb{R}^n, \tag{10}$$

where

$$\begin{aligned} \bar{A} &= \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ -a_0 & -a_1 & -a_2 & \dots & -a_{n-1} \end{bmatrix}, \\ B &= \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}. \end{aligned} \tag{11}$$

Note that applying to the nonlinear system (8) the nonlinear state feedback

$$u = v - g(x), \tag{12}$$

we obtain the linear closed-loop system described by Eqn. (10).

The coefficients $a_k, k = 0, 1, \dots, n-1$, can be chosen so that the linear system (10) is asymptotically stable.

For all values of the coefficients $a_k, k = 0, 1, \dots, n-2$, the pair (11) is controllable since (Jakubczyk and Respondek, 1980; Isidori, 1989)

$$\operatorname{rank}[\, B \quad \bar{A}B \quad \dots \quad \bar{A}^{n-1}B \,] = n. \tag{13}$$

Note that the linear system (10) with (11) is positive if and only if $a_k = 0, k = 0, 1, \dots, n-1$. In this case, the linear system is unstable.

Therefore, the following results have been proven.

Theorem 2. *The nonlinear system (1) can be linearized by the nonlinear state feedback (12) and for a suitable choice of the coefficients* $a_k, k = 0, 1, \dots, n-1$*, the linear closed-loop system (10) is asymptotically stable and controllable.*

Theorem 3. *The nonlinear system (1) can be linearized by the nonlinear state feedback (12), so that the closed-loop system (10) for* $a_k = 0, k = 0, 1, \dots, n-1$*, is positive but unstable.*

4. Example

Consider the nonlinear system described by the equations

$$\dot{x} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} x_2 + x_1^2 \\ x_3 + x_1 x_2 \\ x_2 x_3 + u \end{bmatrix}, \quad x(0) \in \mathbb{R}_+^3. \tag{14}$$

The system (14) is positive since Eqn. (14) satisfies the conditions (3) and $u = u(t) \in \mathbb{R}_+$, $t \geq 0$. In this case, the new state variables $z_k, k = 1, 2, 3$, are defined as follows:

$$z = \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 + x_1^2 \\ x_3 + 3x_1 x_2 + 2x_1^3 \end{bmatrix} = \phi(x) \tag{15}$$

and

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} z_1 \\ z_2 - z_1^2 \\ z_3 - 3z_1 z_2 + z_1^3 \end{bmatrix}. \tag{16}$$

The nonlinear system (14) in the new state variables is described by Eqn. (17).

To linearize the nonlinear system (17), we apply the nonlinear state feedback (12) of the form

$$\begin{aligned} u &= v - g(z) \\ &= v - a_0 z_1 - a_1 z_2 - a_2 z_3 + z_1^5 + 2z_1^4 - 4z_1^3 z_2 \\ &\quad + 3z_1^2 z_2 + z_1^2 z_3 - 3z_2^2 + 3z_1 z_2^2 \\ &\quad - 3z_1 z_3 - z_2 z_3, \end{aligned} \tag{18}$$

and we obtain the linear system (10) with

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_0 & -a_1 & -a_2 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}. \tag{19}$$

The linear system is controllable for all values of the coefficients $a_k, k = 0, 1, 2$, and it is asymptotically stable if and only if $a_k > 0, k = 0, 1, 2$, and $a_1 a_2 > a_0$.

The linear system (10) with (19) is positive if and only if $a_k = 0, k = 0, 1$ since in this case

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \in M_3. \tag{20}$$

In this case the linear system is unstable.

5. Concluding remarks

The positivity and linearization of a class of nonlinear systems by nonlinear state feedbacks were addressed. Necessary and sufficient conditions for the positivity of the class of nonlinear systems (Theorem 1) were established. It was shown that the nonlinear systems can be linearized by nonlinear feedbacks, so that the

$$\dot{z} = \begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \\ \dot{z}_3 \end{bmatrix} = \begin{bmatrix} z_2 \\ z_3 \\ -z_1^5 - 2z_1^4 + 4z_1^3 z_2 - 3z_1^2 z_2 - z_1^2 z_3 + 3z_2^2 - 3z_1 z_2^2 + 3z_1 z_3 + z_2 z_3 + u \end{bmatrix}. \qquad (17)$$

linear close-loop system is asymptotically stable and controllable (Theorem 2) and positive but unstable (Theorem 3). The discussion was illustrated by an example. An open problem is the extension of these deliberations to fractional nonlinear systems.

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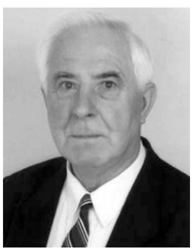

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