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STABILIZED MODEL REDUCTION FOR NONLINEAR DYNAMICAL SYSTEMS THROUGH A CONTRACTIVITY–PRESERVING FRAMEWORK

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This work develops a technique for constructing a reduced-order system that not only has low computational complexity, but also maintains the stability of the original nonlinear dynamical system. The proposed framework is designed to preserve the contractivity of the vector field in the original system, which can further guarantee stability preservation, as well as provide an error bound for the approximated equilibrium solution of the resulting reduced system. This technique employs a low-dimensional basis from proper orthogonal decomposition to optimally capture the dominant dynamics of the original system, and modifies the discrete empirical interpolation method by enforcing certain structure for the nonlinear approximation. The efficiency and accuracy of the proposed method are illustrated through numerical tests on a nonlinear reaction diffusion problem.

Keywords: model order reduction, contractivity, ordinary differential equations, partial differential equations, proper orthogonal decomposition, discrete empirical interpolation method.

1. Introduction

Numerical simulations of many natural phenomena described by nonlinear differential equations can lead to dynamical systems with very large spatial dimensions when standard discretization schemes are applied. To reduce the computational cost for solving each of these large-scale systems, *model reduction* methods can be used to produce a relatively low dimensional model that still provides an accurate solution of the original system. In general, the accuracy of a given model reduction technique is evaluated through certain error measurements when compared with some known reference solutions. Besides considering these approximation errors, this work aims to preserve a fundamental behavior of the original system, which will be done through contraction analysis (Lohmiller and Slotine, 1998). The contraction property can ensure not only the stability of dynamical systems, but also the existence and uniqueness of the solutions, as well as provide an error bound for the approximated equilibrium solution (Söderlind, 2006).

The notion of contractivity for dynamical systems was initially introduced by Lohmiller and Slotine (1998). For historical notes on contractivity theory, see, e.g., the works of Jouffroy (2005) or Lohmiller and Slotine (2000b). The concept of contractivity has been used to analyze important properties of nonlinear dynamical systems in many applications, such as nonlinear control problems (Lohmiller and Slotine, 2000a; 2000b; Habibi *et al.*, 2008), nonlinear stochastic dynamical systems (Benda, 1998; Peigné and Woess, 2011; Pham *et al.*, 2009), dynamical systems represented by Gaussian mixture regression (Blocher *et al.*, 2017), and fractional-order dynamical systems (Wang and Xiao, 2015). Contraction theory has been extended to piecewise smooth dynamical systems (Russo and di Bernardo, 2011) and Riemannian manifolds (Simpson-Porco and Bullo, 2014). A more comprehensive review of contraction theory can be found in, e.g., the works of Jouffroy (2005), Sontag (2010) or Aminzare and Sontag (2014). In this work, contraction analysis will be applied in the context of projection-based nonlinear model reduction.

One of the most popular model reduction methods that preserve the contraction property is a projection-based approach using proper orthogonal decomposition (POD) with the Galerkin projection, i.e., the POD-Galerkin or POD method. This method is successful in substantially reducing the number of state variables. The notion of POD is based on the singular value decomposition (SVD) and it has been used in numerous applications (e.g., Berkooz *et al*., 1993; Lanata and Grosso, 2006; Kunisch and Volkwein, 2010; Schenone, 2014; Gurka *et al*., 2006; Sukuntee and Chaturantabut, 2019; Intawichai and Chaturantabut, 2020). However, for nonlinear systems, the computational complexity of the POD-Galerkin approach generally still depends on the high dimension of the original full-order system since it requires to compute orthogonal projection of nonlinear terms.

To avoid this inefficiency, new approaches have been proposed to improve the POD-Galerkin method for nonlinear systems. These approaches include the trajectory piecewise-linear (TPWL) method (Rewieński, 2003; Rewienski and White, 2006), a reference trajectory based mode control strategy for discrete time dynamical system (Bartoszewicz and Adamiak, 2019), a heuristic algorithm based on linear matrix inequalities (LMIs) (Sanjuan *et al*., 2019), missing point estimation (MPE) (Astrid, 2004), and discrete empirical interpolation (DEIM) (Chaturantabut and Sorensen, 2010).

The TPWL approach is based on estimating a nonlinear function by using linearized approximation constructed from the existing information of the original full-order system. It has been used in many applications, especially in circuit simulations (Rewienski and White, 2001; 2003; 2006). However, not all nonlinear functions can be accurately estimated by linearized approximations.

MPE can reduce the complexity of the POD-Galerkin reduced system by considering certain selected equations in the discretized system. This approach was further extended in the form of a special inner product (Astrid *et al*., 2008). DEIM can be viewed as an improvement of MPE by combining oblique projection with interpolatory approximation.

The interpolated indices are selected based on a greedy algorithm proposed by Barrault *et al*. (2004) for the empirical interpolation method (EIM), which was introduced in a function space setting for the finite element framework with the projection basis obtained directly from snapshot solutions. An error bound for the DEIM approximation shown by Chaturantabut and Sorensen (2010) implies that it is nearly as accurate as the optimal POD approximation. DEIM has been successfully used with the POD method for constructing reduced systems in many recent works, such as in neural modeling (Kellems *et al*., 2010), subsurface flows (Ştefănescu and Navon, 2013; Ghasemi *et al*., 2015; Stanko *et al*., 2016; Chaturantabut, 2017; Isoz, 2019; Sukuntee and Chaturantabut, 2020), coupled circuit-device systems (Hinze *et al*., 2012), and solid mechanics problems (Ghavamian *et al*., 2017). A detailed error analysis of the POD-DEIM approach can be found in the works of Chaturantabut and Sorensen (2012) as well as Wirtz *et al*. (2014).

Despite the success of the POD-DEIM approach in various applications, it still cannot be proved theoretically to preserve stability and other fundamental properties of the original systems. In fact, it will guarantee stability in the sense of contractivity analysis only under certain conditions, as shown later in this work.

The existing model reduction methods mainly focus on preserving the system properties for only some special classes of nonlinear dynamical systems, for example, the framework for preserving the Lagrangian structure of the nonlinear mechanical systems was introduced by Carlberg *et al*. (2015), and the approach for preserving the nonlinear port-Hamiltonian structure was proposed by Chaturantabut *et al*. (2016). For general cases, only the stability for linearized systems was considered by Hochman *et al*. (2011).

This work focuses on preserving stability for general nonlinear systems without requiring linearization. In particular, this work derives a contractivity-preserving framework for nonlinear vector fields, which will be shown to maintain important behaviors of dynamical systems, such as exponential stability, the existence and uniqueness of the solution, and convergence of the perturbed equilibrium. The proposed framework applies the concept of an interpolatory projection-based nonlinear model reduction approach using DEIM with some structured form of the approximated nonlinear term.

This work is organized as follows. First, a general form of nonlinear differential equations and fundamental notions of contractivity are introduced in Section 2. Two projection-based model reduction methods, POD and POD-DEIM approaches, are reviewed in Section 3. Based on these approaches, Section 4 presents the derivation of a model reduction framework that preserves the contractivity of vector field from the original dynamical system. The contractivity is shown to further imply the stability of the solution, as well as can be used to obtain an error bound for a perturbed equilibrium solution. This section also investigates the contractivity property of the existing POD and POD-DEIM techniques. It will be shown that POD reduced systems always preserve the contractivity of the original system, but POD-DEIM systems do not. The conditions under which the POD-DEIM approach preserves the contractivity property are discussed at the end of Section 4. In Section 5, two numerical tests are performed on a nonlinear reaction-diffusion problem to demonstrate the efficiency of the proposed framework. The summary of this work and some final remarks are discussed in Section 6.

2. Problem formulation and contractivity

This section provides some theoretical background required for deriving a model reduction scheme that preserves the contractivity of nonlinear dynamical systems. The desired form of the system structure to be preserved will be discussed together with its significance.

Consider the system of nonlinear ordinary differential equations (ODEs) of the form:

$$\frac{\mathrm{d}\mathbf{y}}{\mathrm{d}t} = \mathbf{F}(t, \mathbf{y}), \qquad \mathbf{y}(0) = \mathbf{y}_0, \tag{1}$$

where $\mathbf{y} = \mathbf{y}(t)$ is an n dimensional state variable at some time $t \geq 0$ and $\mathbf{F} : [0, \infty) \times Y \to \mathbb{R}^n$ is a differentiable nonlinear vector-valued function, $Y \subseteq \mathbb{R}^n$ with the Jacobian given by $J_{\mathbf{F}}(t, \mathbf{y}) = \frac{\partial \mathbf{F}}{\partial \mathbf{y}}(t, \mathbf{y})$.

We are interested in constructing a model reduction that preserves stability properties of the original system. The standard stability properties are generally analyzed through a Lyapunov-based approach. However, a main difficulty for this standard analysis often arises as it requires equilibrium points to be specified in advance. This work considers an alternative stability criterion using *contraction* analysis, which is generally easier to analyze but stronger than the standard one. In particular, while standard nonlinear stability has to be analyzed with respect to an equilibrium solution, contraction is concerned with the behavior of system trajectories with respect to each other and does not require the prior knowledge of the steady-state solution. In this work, contraction analysis will be mainly applied to the vector field defining the dynamical system.

2.1. Logarithmic norm and logarithmic Lipschitz constants.

We first consider the *logarithmic norm*, introduced independently by Dahlquist (1959) and Lozinskii (1958). The definition of *logarithmic norm* is given below in a special case of Euclidean space.

Definition 1. (*Logarithmic norm*) Let $\mathbf{A} \in \mathbb{R}^{n \times n}$ be a constant matrix. The associated matrix measure, called *logarithmic norm* is defined as

$$\mu[\mathbf{A}] = \lim_{h \to 0^+} \frac{\|I + h\mathbf{A}\| - 1}{h}, \tag{2}$$

where $\|\cdot\|$ is the standard Euclidean norm.

In the above definition, $\|\cdot\|$ can be any norm. When $\|\cdot\|$ is the Euclidean norm, it can be shown that (Söderlind, 2006), $\mu[\mathbf{A}]$ is the maximum eigenvalue of the symmetric part of $\mathbf{A}$, i.e.,

$$\mu[\mathbf{A}] = \lambda_{\max}\left(\frac{\mathbf{A} + \mathbf{A}^T}{2}\right), \tag{3}$$

where $\lambda_{\max}(\cdot)$ stands for the maximum eigenvalue of the argument. Equivalently, it can also be shown that, for any induced norm in Hilbert space,

$$\mu[\mathbf{A}] = \sup_{\mathbf{u} \neq 0} \frac{\operatorname{Re}\langle \mathbf{u}, \mathbf{A}\mathbf{u}\rangle}{\langle \mathbf{u}, \mathbf{u}\rangle}, \tag{4}$$

where $\|\mathbf{u}\|^2 = \langle \mathbf{u}, \mathbf{u}\rangle$, $\mathbf{u} \in \mathbb{R}^n$. The concept of logarithmic norm has been previously used to analyze the convergence of solutions from ordinary differential equations (Hairer *et al.*, 1993; Banasiak, 2020). The notion of logarithmic norm has been extended to a general nonlinear operator in Banach spaces by introducing the notion of *logarithmic Lipschitz constants* (Söderlind, 2006). The definition and some elementary properties of logarithmic Lipschitz constants are given below.

Definition 2. (*Least upper bound/greatest lower bound Lipschitz constants*) Let $(X, \|\cdot\|_X)$ be a normed space and $F : Y \to X$ be a function where $Y \subseteq X$. The *least upper bound (lub)* and the *greatest lower bound (glb) Lipschitz constants* of F induced by the norm $\|\cdot\|_X$ on Y are defined, respectively, by

$$L_{Y,X}[F] = \sup_{u \neq v \in Y} \frac{\|F(u) - F(v)\|_X}{\|u - v\|_X}, \tag{5}$$

$$\ell_{Y,X}[F] = \inf_{u \neq v \in Y} \frac{\|F(u) - F(v)\|_X}{\|u - v\|_X}. \tag{6}$$

The *least upper bound (lub)* and the *greatest lower bound (glb) logarithmic Lipschitz constants* of F induced by the norm $\|\cdot\|_X$ on Y are defined by

$$M_{Y,X}[F] = \lim_{h \to 0^+} \frac{L_{Y,X}[I + hF] - 1}{h}, \tag{7}$$

$$m_{Y,X}[F] = \lim_{h \to 0^-} \frac{L_{Y,X}[I + hF] - 1}{h}. \tag{8}$$

Note that this work considers the setting for systems of ODEs with $X = Y \subseteq \mathbb{R}^n$ and will use the notation $L_{X,X}[\cdot] = L[\cdot]$, $\ell_{X,X}[\cdot] = \ell[\cdot]$ and $M_{X,X}[\cdot] = M[\cdot]$, $m_{X,X}[\cdot] = m[\cdot]$. Moreover, the Euclidean norm will be used for $\|\cdot\|_X$, which will be simply denoted as $\|\cdot\|$. In this case, it can be shown (Söderlind, 2006) that $M[\cdot] = m[\cdot]$ and for any $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$,

$$M[F] = \sup_{\mathbf{u} \neq \mathbf{v}} \frac{\langle \mathbf{u} - \mathbf{v}, F(\mathbf{u}) - F(\mathbf{v})\rangle}{\|\mathbf{u} - \mathbf{v}\|^2} \tag{9}$$

$$= \sup_{\mathbf{u} \neq \mathbf{v}} \frac{(\mathbf{u} - \mathbf{v})^T (F(\mathbf{u}) - F(\mathbf{v}))}{\|\mathbf{u} - \mathbf{v}\|^2}. \tag{10}$$

Note that, from (4), when $F = \mathbf{A} \in \mathbb{R}^{n \times n}$, $\mu[\mathbf{A}] = M[\mathbf{A}]$ because (4) can be written as

$$\mu[\mathbf{A}] = \sup_{\mathbf{u} \neq \mathbf{v}} \frac{\operatorname{Re}\langle \mathbf{u} - \mathbf{v}, \mathbf{A}\mathbf{u} - \mathbf{A}\mathbf{v}\rangle}{\langle \mathbf{u} - \mathbf{v}, \mathbf{u} - \mathbf{v}\rangle},$$

for $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$.

Lemma 1. (Aminzare and Sontag, 2014; Söderlind, 1986) *Let M be the (lub) logarithmic Lipschitz constant induced by the Euclidean norm on $\mathbb{R}^n$ and $Y \subseteq \mathbb{R}^n$ be a connected set. Then for any Lipschitz and continuously differentiable function $F : Y \to \mathbb{R}^n$, with Jacobian J_F,*

$$\sup_{\mathbf{y}\in Y} \mu[J_F(\mathbf{y})] \leq M[F]. \tag{11}$$

In addition, if Y is convex, then

$$\sup_{\mathbf{y}\in Y} \mu[J_F(\mathbf{y})] = M[F]. \tag{12}$$

This lemma is useful in practice for estimating or computing $M[F]$ when the Jacobian J_F is known.

2.2. Contractivity. The appropriate definition and related properties of *contractivity* will be presented next for the vector field $\mathbf{F}$ of the differential equation in (1).

Definition 3. (*Infinitesimally contracting*) (Aminzare and Sontag, 2014; Sontag, 2010) The time-dependent vector field $\mathbf{F} : [0, \infty) \times Y \to \mathbb{R}^n$, $Y \subseteq \mathbb{R}^n$, in the system (1), is said to be *infinitesimally contracting* on a set $Y \subseteq \mathbb{R}^n$ with respect to the Euclidean norm if, for some constant $c > 0$,

$$\mu[J_{\mathbf{F}}(t, \mathbf{y})] \leq -c, \qquad \forall \mathbf{y} \in Y, \quad \forall t \geq 0. \tag{13}$$

where $J_{\mathbf{F}}(t, \mathbf{y}) \in \mathbb{R}^{n\times n}$ is the Jacobian of $\mathbf{F}(t, \mathbf{y})$. The constant c is called the *contraction rate*.

Remark 1. For $\mathbf{F} : [0, \infty) \times Y \to \mathbb{R}^n$, $Y \subseteq \mathbb{R}^n$, recall from Lemma 1 that

$$\sup_{\mathbf{y}\in Y} \mu\left[J_{\mathbf{F}}(t, \mathbf{y})\right] \leq M[\mathbf{F}_t], \qquad \forall t \geq 0,$$

where $\mathbf{F}_t(\mathbf{y}) = \mathbf{F}(t, \mathbf{y})$. Therefore, the function $\mathbf{F}$ is *infinitesimally contracting* if

$$\sup_{t\in[0,\infty)} M[\mathbf{F}_t] < 0. \tag{14}$$

In this work, the above stronger condition of being *infinitesimally contracting* given in (14) will be used instead of (13) to make it more convenient for applying to general nonlinear functions when deriving a contractivity-preserving model reduction approach. For the time-independent function $\mathbf{F}$, i.e., $\mathbf{F}(t, \mathbf{y}) = \mathbf{F}(\mathbf{y}), \forall t \in [0, \infty)$, which will be considered mainly in this work, the condition (14) simply becomes $M[\mathbf{F}] < 0$. It is important to note that *infinitesimal contractivity* implies *global contractivity*, as shown, e.g., by Aminzare and Sontag (2013).

Theorem 1. (Aminzare and Sontag, 2013; 2014) *Let $\|\cdot\|$ be the Euclidean norm and $\mathbf{F} : [0, \infty) \times Y \to X$ be (globally) Lipschitz and continuously differentiable function, where $Y \subseteq X = \mathbb{R}^n$. Suppose $\mathbf{y}$ and $\hat{\mathbf{y}}$ are the solutions of $\frac{\mathrm{d}\mathbf{y}}{\mathrm{d}t} = \mathbf{F}(t, \mathbf{y})$, with initial conditions $\mathbf{y}(0) = \mathbf{y}_0$ and $\mathbf{y}(0) = \hat{\mathbf{y}}_0$, respectively. Then, for*

$$k := \sup_{t\in[0,\infty)} M[\mathbf{F}_t], \tag{15}$$

$$\|\mathbf{y}(t) - \hat{\mathbf{y}}(t)\| \leq e^{kt}\|\mathbf{y}_0 - \hat{\mathbf{y}}_0\|, \quad \forall t \geq 0. \tag{16}$$

where $\mathbf{F}_t(\mathbf{y}) = \mathbf{F}(t, \mathbf{y})$ as defined in the previous remark.

From Theorem 1, when $\mathbf{F}$ is infinitesimally contracting, i.e., $k < 0$, the trajectories globally and exponentially converge to each other. In the remaining parts of this paper, the *contractivity* of a function will refer to the condition (14), which will imply both *infinitesimal contractivity* and *global contractivity*.

To see some effects of contractivity on the behaviors of dynamical systems with nonlinear vector field $\mathbf{F}$, consider a simple mathematical example. Without any loss of generality, the system of differential equations (1) is assumed to be autonomous for notational convenience. Consider the two systems of differential equations:

$$\frac{\mathrm{d}\mathbf{y}}{\mathrm{d}t} = \mathbf{F}(\mathbf{y}), \qquad \mathbf{y}(0) = \mathbf{y}_0, \tag{17}$$

$$\frac{\mathrm{d}\hat{\mathbf{y}}}{\mathrm{d}t} = \mathbf{F}(\hat{\mathbf{y}}) + \mathbf{p}(t), \qquad \hat{\mathbf{y}}(0) = \hat{\mathbf{y}}_0. \tag{18}$$

The system (18) can be viewed as a perturbed system of the system (17). Let $\mathbf{E}(t) = \hat{\mathbf{y}}(t) - \mathbf{y}(t)$ be the difference of the solutions from these two systems. Then $\dot{\mathbf{E}}(t) = \dot{\hat{\mathbf{y}}}(t) - \dot{\mathbf{y}}(t) = \mathbf{F}(t, \hat{\mathbf{y}}) - \mathbf{F}(t, \mathbf{y}) + \mathbf{p}(t)$. By using the identity (9) and $\|\mathbf{E}\|\frac{\mathrm{d}}{\mathrm{d}t}\|\mathbf{E}\| = \frac{1}{2}\|\mathbf{E}\|^2 = \mathbf{E}^T\dot{\mathbf{E}} = \langle \mathbf{E}, \dot{\mathbf{E}}\rangle$, we have

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\|\mathbf{E}\| &= \frac{1}{\|\mathbf{E}\|}\langle \mathbf{E}, \dot{\mathbf{E}}\rangle \\
&= \frac{1}{\|\mathbf{E}\|}\langle \mathbf{E}, \mathbf{F}(t, \hat{\mathbf{y}}) - \mathbf{F}(t, \mathbf{y}) + \mathbf{p}(t)\rangle, \\
\frac{\mathrm{d}}{\mathrm{d}t}\|\hat{\mathbf{y}} - \mathbf{y}\| &= \frac{1}{\|\hat{\mathbf{y}} - \mathbf{y}\|}\langle \hat{\mathbf{y}} - \mathbf{y}, \mathbf{F}(t, \hat{\mathbf{y}}) - \mathbf{F}(t, \mathbf{y})\rangle \\
&\quad + \frac{1}{\|\mathbf{E}\|}\langle \mathbf{E}, \mathbf{p}(t)\rangle \\
&\leq M[\mathbf{F}]\|\hat{\mathbf{y}} - \mathbf{y}\| + \|\mathbf{p}(t)\|, \\
\frac{\mathrm{d}}{\mathrm{d}t}\|\mathbf{E}\| &\leq M[\mathbf{F}]\|\mathbf{E}\| + \|\mathbf{p}(t)\|,
\end{aligned}$$

which implies that, for $t \geq 0$,

$$\begin{aligned}
\|\mathbf{E}(t)\| &\leq \|\mathbf{E}(0)\|\, e^{M[\mathbf{F}]t} \\
&\quad + \int_0^t \|\mathbf{p}(\tau)\|\, e^{(t-\tau)M[\mathbf{F}]}\,\mathrm{d}\tau.
\end{aligned} \tag{19}$$

The above bound illustrates the effects of the *logarithmic Lipschitz constant* on certain system's properties, such as stability and perturbation. As shown by Söderlind (2006), two fundamental cases should be considered for the error $\|\mathbf{E}(t)\| = \|\hat{\mathbf{y}}(t) - \mathbf{y}(t)\|$ from (19).

Case 1. When $\mathbf{p}(t) = 0$, the bound in (19) gives

$$\|\hat{\mathbf{y}}(t) - \mathbf{y}(t)\| \leq e^{M[\mathbf{F}]t}\|\hat{\mathbf{y}}(0) - \mathbf{y}(0)\|.$$

When $\mathbf{F}$ is infinitesimally contracting, i.e., $M[\mathbf{F}] < 0$, the solution is exponentially stable.

Case 2. When $\mathbf{E}(0) = 0$, i.e., the initial conditions $\mathbf{y}_0$ and $\hat{\mathbf{y}}_0$ are the same, the bound in (19) gives

$$\|\hat{\mathbf{y}}(t) - \mathbf{y}(t)\| \leq \frac{e^{tM[\mathbf{F}]} - 1}{M[\mathbf{F}]} \max_{t\in[0,\infty)} \|\mathbf{p}(t)\|,$$

by using a straightforward integration. Notice that when $\mathbf{F}$ is infinitesimally contracting, i.e., $M[\mathbf{F}] < 0$, we have $e^{tM[\mathbf{F}]} \in (0, 1)$ and the bound becomes $\|\hat{\mathbf{y}}(t) - \mathbf{y}(t)\| \leq \frac{-1}{M[\mathbf{F}]} \max_{t\in[0,\infty)} \|\mathbf{p}(t)\|$, which implies that $\hat{\mathbf{y}}(t) \to \mathbf{y}(t)$ as $\max_{t\in[0,\infty)} \|\mathbf{p}(t)\| \to 0$.

Based on the discussion above, it follows that it is essential to maintain the contractivity of the vector field when constructing the approximate low-dimensional system, so that the fundamental behaviors of the original system are preserved.

3. Model order reduction

In order to derive a contractivity-preserving reduced-order modeling, this section will first consider a well-known method called proper orthogonal decomposition (POD) and its combination with the discrete empirical interpolation method (DEIM).

Recall the nonlinear differential equation (1) in the form of the autonomous system

$$\frac{\mathrm{d}\mathbf{y}}{\mathrm{d}t} = \mathbf{F}(\mathbf{y}), \qquad \mathbf{y}(0) = \mathbf{y}_0, \tag{20}$$

where $\mathbf{y} : [0,\infty) \to \mathbb{R}^n$ is the state variable and $\mathbf{F} : Y \subseteq \mathbb{R}^n \to \mathbb{R}^n$ is the nonlinear vector field. The projection-based model reduction method can construct a reduced-order system by projecting (20) onto a low dimensional subspace. Let $\mathbf{V} \in \mathbb{R}^{n\times k}$ be a matrix whose columns form a set of an orthonormal basis of dimension k, where $k \leq n$. Then we can approximate the state variable $\mathbf{y}(t)$ in the space spanned by the columns of $\mathbf{V}$ in the form $\mathbf{y}(t) \approx \mathbf{V}\tilde{\mathbf{y}}(t)$, where $\tilde{\mathbf{y}}(t) \in \mathbb{R}^k$. After substituting this approximation into (20) and applying the Galerkin projection to obtain the smallest residual error in the direction of $\mathrm{span}\{\mathbf{V}\}$, we have $\mathbf{V}^T \left[\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{V}\tilde{\mathbf{y}}(t) - \mathbf{F}(\mathbf{V}\tilde{\mathbf{y}}(t))\right] = 0$. Note that $\mathrm{span}\{\mathbf{V}\}$ is defined to be the space spanned by the columns of matrix $\mathbf{V}$. Since the columns of $\mathbf{V}$ are orthonormal, the projected reduced system is of the form

$$\frac{\mathrm{d}}{\mathrm{d}t}\tilde{\mathbf{y}}(t) = \mathbf{V}^T\mathbf{F}(\mathbf{V}\tilde{\mathbf{y}}(t)), \quad \tilde{\mathbf{y}}(0) = \mathbf{V}^T\mathbf{y}_0. \tag{21}$$

In this setting, $\mathbf{V}$ can be obtained from any orthogonal basis. However, to get an accurate approximation from this reduced system, proper orthogonal decomposition (POD) will be used to construct this basis. POD can optimally extract the dominant characteristics from any given system of interest as shown in its definition (22).

Proper orthogonal decomposition (POD) is also known by other names, for example, the Karhunen-Loève decomposition (KLD), principal component analysis (PCA), or singular value decomposition (SVD). POD has been used with the Galerkin projection in many applications to reduce the number of variables of large-scaled discretized systems (e.g., Berkooz *et al.*, 1993; Lanata and Grosso, 2006; Kunisch and Volkwein, 2010; Schenone, 2014; Gurka *et al.*, 2006). One of the most important properties of POD is that it can construct an approximation that minimizes the error in the 2-norm for a given fixed basis rank k. POD also can be obtained by using singular value decomposition (SVD) as discussed next.

Definition 4. (*POD basis*) *(Volkwein, 2008)* Let $\mathbf{Y} = [\mathbf{y}_1, \ldots, \mathbf{y}_{n_s}] \in \mathbb{R}^{n\times n_s}$ be a snapshot matrix with rank $r \leq \min\{n, n_s\}$. The POD basis of dimension k, where $k \leq r$, is the solution to the following optimization problem:

$$\min_{\mathbf{\Phi}_k\in\mathbb{R}^{n\times k}} \sum_{j=1}^{n_s} \|\mathbf{y}_j - \mathbf{\Phi}_k\mathbf{\Phi}_k^T\mathbf{y}_j\|_2^2 \tag{22}$$

such that

$$\mathbf{\Phi}_k^T\mathbf{\Phi}_k = \mathbf{I}_k$$

where $\mathbf{I}_k \in \mathbb{R}^{k\times k}$ is the identity matrix.

It can be shown (Volkwein, 2008) that the POD basis defined above can be obtained from the left singular vector of the snapshot matrix $\mathbf{Y}$. Let $\mathbf{Y} = \hat{\mathbf{U}}\mathbf{\Sigma}\mathbf{Z}^T$ be the singular value decomposition of $\mathbf{Y}$, where matrices $\hat{\mathbf{U}} = [\mathbf{u}_1, \ldots, \mathbf{u}_r] \in \mathbb{R}^{n\times r}$ and $\hat{\mathbf{Z}} = [\mathbf{z}_1, \ldots, \mathbf{z}_r] \in \mathbb{R}^{n_s\times r}$ are matrices with orthogonal columns and $\mathbf{\Sigma} = \mathrm{diag}(\sigma_1, \ldots, \sigma_r) \in \mathbb{R}^{r\times r}$ is a diagonal matrix with $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_r > 0$. Then the POD basis matrix $\mathbf{V} \in \mathbb{R}^{n\times k}$ of dimension k, where $k \leq r$, is given by $\mathbf{V} = [\mathbf{u}_1, \ldots, \mathbf{u}_k]$, which consists of the first k columns of $\hat{\mathbf{U}}$. It is well-known (Volkwein, 2008) that

$$\sum_{j=1}^{n_s} \|\mathbf{y}_j - \mathbf{V}\mathbf{V}^T\mathbf{y}_j\|_2^2 = \sum_{\ell=k+1}^{r} \sigma_\ell^2, \tag{23}$$

which is the sum of the neglected singular values $\sigma_{k+1}, \ldots, \sigma_r$ from the SVD of $\mathbf{Y}$.

When the matrix $\mathbf{V}$ is obtained from POD, the system (21) is called the POD reduced system or the POD-Galerkin reduced system. Although the

POD-Galerkin approach can reduce the number of unknowns of the full-order system, it may not be able to reduce the complexity for computing the projected nonlinear term $\mathbf{V}^T\mathbf{F}(\mathbf{V}\tilde{\mathbf{y}}(t))$ in (21). To handle this complexity problem, POD will be used with the discrete empirical interpolation method (DEIM) (Chaturantabut and Sorensen, 2010).

The DEIM approximates the nonlinear function $\mathbf{F}(\mathbf{y})$ by projecting it onto the space spanned by the columns of a basis matrix $\mathbf{U} \in \mathbb{R}^{n\times m}$ of rank $m \leq n$. The matrix $\mathbf{U}$ can be constructed from the POD basis of the nonlinear snapshot matrix $[\mathbf{F}(\mathbf{y}_1), \ldots, \mathbf{F}(\mathbf{y}_{n_s})]$, where $\mathbf{y}_i \cong \mathbf{y}(t_i)$. This DEIM approximation for the POD vector field is therefore in the form of $\mathbf{F}(\mathbf{V}\tilde{\mathbf{y}}(t)) \approx \mathbf{U}\mathbf{c}(t)$, for some vector $\mathbf{c}(t)$ in $\mathbb{R}^m$. In order to specify $\mathbf{c}(t)$, a greedy selection procedure given in Algorithm 1 is used to select m interpolated row indices of the interpolation approximation. That is, let $\wp_1, \ldots, \wp_m$ be the interpolation indices from Algorithm 1 corresponding to the input basis set from $\mathbf{U}$ and let $\mathbf{P} = [\mathbf{e}_{\wp_1}, \ldots, \mathbf{e}_{\wp_m}] \in \mathbb{R}^{n\times m}$, where $\mathbf{e}_{\wp_i} = [0, \ldots, 0, 1, 0, \ldots, 0]^T \in \mathbb{R}^n$ is the $\wp_i$-th column of the identity matrix $\mathbf{I}_n \in \mathbb{R}^{n\times n}$, for $i = 1, \ldots, m$.

Since it has been shown by Chaturantabut and Sorensen (2010) that $\mathbf{P}^T\mathbf{U}$ is nonsingular, the vector $\mathbf{c}(t)$ can be uniquely determined from

$$\mathbf{P}^T\mathbf{F}(\mathbf{V}\tilde{\mathbf{y}}(t)) = (\mathbf{P}^T\mathbf{U})\mathbf{c}(t), \tag{24}$$

which gives a closed-form formula $\mathbf{c}(t) = (\mathbf{P}^T\mathbf{U})^{-1}\mathbf{P}^T\mathbf{F}(\mathbf{V}\tilde{\mathbf{y}}(t))$. Therefore, the approximation is given by

$$\mathbf{F}(\mathbf{V}\tilde{\mathbf{y}}(t)) = \mathbf{U}(\mathbf{P}^T\mathbf{U})^{-1}\underbrace{\mathbf{P}^T\mathbf{F}(\mathbf{V}\tilde{\mathbf{y}}(t))}_{m\times 1}. \tag{25}$$

In the case when the nonlinear function $\mathbf{F}$ is componentwise, i.e., $[\mathbf{F}(\mathbf{z})]_i = F_i(z_i)$, for $\mathbf{z} \in Y \subseteq \mathbb{R}^n$, $i = 1, \ldots, n$ and $F_i(\cdot)$ is the i-th component of $\mathbf{F}(\cdot)$, we have

$$\mathbf{F}(\mathbf{V}\tilde{\mathbf{y}}(t)) = \mathbf{U}(\mathbf{P}^T\mathbf{U})^{-1}\underbrace{\mathbf{F}(\mathbf{P}^T\mathbf{V}\tilde{\mathbf{y}}(t))}_{m\times 1}. \tag{26}$$

The assumption on the componentwise function might seem to be rather strong. However, many partial differential equations have nonlinear terms that correspond to componentwise functions in their discretized systems, e.g., reaction terms in most flow models. Also, this assumption can be avoided by using the structure of the dependence in state variables in each component of the nonlinear term (e.g., incorporating a technique that uses a computational graph based on "Automatic Differentiation by OverLoading in C++ (ADOL-C)" (Walther *et al.*, 2003)).

Note that pre-multiplying by $\mathbf{P}^T$ in (24) is equivalent to extracting the m rows corresponding to

Algorithm 1. Algorithm to create interpolation indices from DEIM.

INPUT: A set of linearly independent vectors $\{\mathbf{u}_\ell\}_{\ell=1}^m \subset \mathbb{R}^n$.

OUTPUT: An index set $\vec{\wp} = [\wp_1, \ldots, \wp_m]^T \in \mathbb{R}^m$.

$[|\rho|, \wp_1] = \max\{|\mathbf{u}_1|\}$
$\mathbf{U} = [\mathbf{u}_1], \mathbf{P} = [\mathbf{e}_{\wp_1}], \vec{\wp} = [\wp_1]$;
for $\ell \leftarrow 2$ to m **do**
  Solve $(\mathbf{P}^T\mathbf{U})\mathbf{c} = \mathbf{P}^T\mathbf{u}_\ell$;
  $\mathbf{r} = \mathbf{u}_\ell - \mathbf{U}\mathbf{c}$ $[|\rho|, \wp_\ell] = \max\{|\mathbf{r}|\}$
  $\mathbf{U} \leftarrow [\mathbf{U} \quad \mathbf{u}_\ell], \mathbf{P} \leftarrow [\mathbf{P} \quad \mathbf{e}_{\wp_\ell}], \vec{\wp} \leftarrow \begin{bmatrix} \vec{\wp} \\ \wp_\ell \end{bmatrix}$
end

the interpolation indices $\wp_1, \ldots, \wp_m$, and there is no actual matrix multiplication required for $\mathbf{P}^T\mathbf{F}(\mathbf{V}\tilde{\mathbf{y}}(t))$. The procedure for selecting these indices is shown in Algorithm 1. It chooses each index by aiming to minimize the residual error $\mathbf{r} = \mathbf{u}_\ell - \mathbf{U}\mathbf{c}$ in each iteration ℓ. Finally, the POD-DEIM reduced system can be written in the following two equivalent forms:

$$\dot{\tilde{\mathbf{y}}} = \mathbf{V}^T\mathbf{U}(\mathbf{P}^T\mathbf{U})^{-1}\mathbf{P}^T\mathbf{F}(\mathbf{V}\tilde{\mathbf{y}}), \quad \tilde{\mathbf{y}}(0) = \mathbf{V}^T\mathbf{y}_0, \tag{27}$$

$$\dot{\tilde{\mathbf{y}}} = \mathbf{V}^T\mathbb{P}\mathbf{F}(\mathbf{V}\tilde{\mathbf{y}}), \quad \tilde{\mathbf{y}}(0) = \mathbf{V}^T\mathbf{y}_0, \tag{28}$$

where $\mathbb{P} := \mathbf{U}(\mathbf{P}^T\mathbf{U})^{-1}\mathbf{P}^T$ is an oblique projector. The first form (27) is generally used in practice by precomputing the term $\mathbf{V}^T\mathbf{U}(\mathbf{P}^T\mathbf{U})^{-1}$. The second form (28) will be used in this work to derive a contractivity-preserving framework.

Although DEIM has been successfully used to obtain accurate low-complexity models in various applications, as can be seen in, e.g., the works of (Kellems *et al.*, 2010; Chaturantabut and Sorensen, 2011; Ştefănescu and Navon, 2013; Feng and Soulaimani, 2007), it cannot be theoretically proved to preserve the stability of original systems through contraction analysis. This work aims to derive a modified form of the POD-DEIM reduced system to overcome this problem.

4. Contractivity-preserving model reduction

Motivated by projection-based model reduction approaches described in the previous section, this section will first propose a general form of the model reduction scheme that preserves the contractivity of nonlinear vector fields in the original systems. The derivation is performed through the Euclidean norm. A specific form that preserves the contractivity will be considered at the end of this section by enforcing certain structure on the modified POD-DEIM reduced system. The contractivity of the existing POD and POD-DEIM approaches will be also investigated. It will be shown that while POD reduced systems always preserve the contractivity, this may not be true for POD-DEIM reduced systems. The conditions under which the POD-DEIM approach preserves this property will be discussed.

4.1. Proposed general form of the contractivity-preserving reduced model. Consider the autonomous differential equation of the form (20). This section proposes a general form of the projection-based model reduction that preserves the contractivity of the original system (20) with respect to the Euclidean norm.

Lemma 2. *Suppose the nonlinear vector field* $\mathbf{F}$ *in (20) is* infinitesimally contracting, *i.e.,* $M[\mathbf{F}] < 0$. *Consider the reduced-order model of (20) in the form*

$$\dot{\tilde{\mathbf{y}}} = \widetilde{\mathbf{F}}(\tilde{\mathbf{y}}), \quad \textit{with} \quad \widetilde{\mathbf{F}}(\tilde{\mathbf{y}}) = \mathbf{V}^T\mathbf{W}\mathbf{F}(\mathbf{W}^T\mathbf{V}\tilde{\mathbf{y}}), \qquad (29)$$

where $\tilde{\mathbf{y}} = \tilde{\mathbf{y}}(t) \in \mathbb{R}^k$, $\mathbf{V} \in \mathbb{R}^{n\times k}$ *has orthonormal columns, for* $k \le n$, $t \ge 0$, *and* $\mathbf{W} \in \mathbb{R}^{n\times n}$ *is a matrix such that* $\mathbf{W}^T\mathbf{V} \in \mathbf{R}^{n\times k}$ *has full column rank, i.e.,* $\mathrm{rank}(\mathbf{W}^T\mathbf{V}) = k$. *Then the nonlinear vector field* $\widetilde{\mathbf{F}}(\tilde{\mathbf{y}})$ *in (29) is also* infinitesimally contracting.

Note that the solution $\mathbf{y}$ of the original full-order system (20) can be approximated by $\mathbf{V}\tilde{\mathbf{y}}$, where $\tilde{\mathbf{y}}$ is the solution from (29). Note also that the matrix $\mathbf{W}$ is introduced in (29) to allow the reduced system to introduce an additional efficient nonlinear complexity reduction, e.g., as explained in Section 3.

Proof. Let $M[\mathbf{F}]$ and $M[\widetilde{\mathbf{F}}]$ be the logarithmic Lipschitz constants of $\mathbf{F}$ and $\widetilde{\mathbf{F}}$, respectively. For $\tilde{\mathbf{u}}, \tilde{\mathbf{v}} \in \mathbb{R}^k$ and $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$, since $\mathbf{F}$ in system (1) is infinitesimally contracting, i.e., $M[\mathbf{F}] < 0$, we have, for $\widetilde{\mathbf{W}} := \mathbf{W}^T\mathbf{V} \in \mathbf{R}^{n\times k}$,

$$\begin{aligned}
M[\widetilde{\mathbf{F}}] &= \sup_{\tilde{\mathbf{u}}\neq\tilde{\mathbf{v}}} \frac{(\tilde{\mathbf{u}}-\tilde{\mathbf{v}})^T(\widetilde{\mathbf{F}}(\tilde{\mathbf{u}})-\widetilde{\mathbf{F}}(\tilde{\mathbf{v}}))}{\|\tilde{\mathbf{u}}-\tilde{\mathbf{v}}\|^2} \\
&= \sup_{\tilde{\mathbf{u}}\neq\tilde{\mathbf{v}}} \frac{(\tilde{\mathbf{u}}-\tilde{\mathbf{v}})^T(\widetilde{\mathbf{W}}^T\mathbf{F}(\widetilde{\mathbf{W}}\tilde{\mathbf{u}})-\widetilde{\mathbf{W}}^T\mathbf{F}(\widetilde{\mathbf{W}}\tilde{\mathbf{v}}))}{\|\tilde{\mathbf{u}}-\tilde{\mathbf{v}}\|^2} \\
&= \frac{1}{K^2}\sup_{\tilde{\mathbf{u}}\neq\tilde{\mathbf{v}}} \frac{(\widetilde{\mathbf{W}}\tilde{\mathbf{u}}-\widetilde{\mathbf{W}}\tilde{\mathbf{v}})^T(\mathbf{F}(\widetilde{\mathbf{W}}\tilde{\mathbf{u}})-\mathbf{F}(\widetilde{\mathbf{W}}\tilde{\mathbf{v}}))}{\|\widetilde{\mathbf{W}}\tilde{\mathbf{u}}-\widetilde{\mathbf{W}}\tilde{\mathbf{v}}\|^2} \\
&\le \frac{1}{K^2}\sup_{\mathbf{u}\neq\mathbf{v}} \frac{(\mathbf{u}-\mathbf{v})^T(\mathbf{F}(\mathbf{u})-\mathbf{F}(\mathbf{v}))}{\|\mathbf{u}-\mathbf{v}\|^2} \\
&= \frac{1}{K^2}M[\mathbf{F}] < 0,
\end{aligned}$$

where K is a positive constant such that $\|\tilde{\mathbf{u}}-\tilde{\mathbf{v}}\| = K\|\widetilde{\mathbf{W}}\tilde{\mathbf{u}}-\widetilde{\mathbf{W}}\tilde{\mathbf{v}}\|$. The assumption that $\widetilde{\mathbf{W}}$ has full column rank guarantees the existence of the constant $K > 0$ and ensures that the denominator $\|\widetilde{\mathbf{W}}\tilde{\mathbf{u}}-\widetilde{\mathbf{W}}\tilde{\mathbf{v}}\|$ is nonzero. Note that when $\widetilde{\mathbf{W}}$ has orthonormal columns, $\|\widetilde{\mathbf{W}}\tilde{\mathbf{u}}-\widetilde{\mathbf{W}}\tilde{\mathbf{v}}\| = \|\widetilde{\mathbf{W}}\|\|\tilde{\mathbf{u}}-\tilde{\mathbf{v}}\|$ and we can use $K = 1/\|\widetilde{W}\|$. Hence, we have $M[\widetilde{\mathbf{F}}] < 0$ and $\widetilde{\mathbf{F}}$ is therefore infinitesimally contracting. ∎

The above result can be extended to guarantee the stability and the existence of the equilibrium solution of the reduced system in the form (29) as discussed in Section 2.2.

Proposition 1. *Suppose the nonlinear vector field* $\mathbf{F}$ *in the full-order system (20) is infinitesimally contracting, i.e.,* $M[\mathbf{F}] < 0$. *Then*

(i) the reduced system (29) preserves the exponential stability of (20) .

(ii) the reduced system (29) has a unique equilibrium $\tilde{\mathbf{y}}_e$, *i.e.,* $\widetilde{\mathbf{F}}(\tilde{\mathbf{y}}_e) = 0$. *Moreover, if* $\mathbf{y}_e$ *is the unique equilibrium solution of (20), then* $\mathbf{y}_e$ *can be approximated by* $\mathbf{V}\tilde{\mathbf{y}}_e$ *with an error bound given by*

$$\|\mathbf{y}_e - \mathbf{V}\tilde{\mathbf{y}}_e\| \le \frac{-\|\mathbf{p}\|}{M[\mathbf{F}]}, \qquad (30)$$

where $\mathbf{p} = \mathbf{F}(\mathbf{V}\tilde{\mathbf{y}}_e)$.

Proof.
(i) This part follows the results of Söderlind (2006) that the solution of the reduced system satisfies $\|\tilde{\mathbf{y}}\| \le e^{M[\widetilde{\mathbf{F}}]t}\|\tilde{\mathbf{y}}(0)\|$ and that $\mathbf{F}$ in system (20) is infinitesimally contracting: $M[\widetilde{\mathbf{F}}] < 0$ for $t \ge 0$ from the previous lemma.
(ii) First note that, after Söderlind (2006), $M[\mathbf{F}] < 0$ implies that the map $\mathbf{F}$ is bijective and there must be a unique solution $\mathbf{y}_e$ such that $\mathbf{F}(\mathbf{y}_e) = 0$. Similarly, from Lemma 2, $M[\mathbf{F}] < 0$ implies $M[\widetilde{\mathbf{F}}] < 0$, which also further gives the existence of the unique solution $\tilde{\mathbf{y}}_e$ such that $\widetilde{\mathbf{F}}(\tilde{\mathbf{y}}_e) = 0$ (Söderlind, 2006). Note that, from the definition of $M[\mathbf{F}]$,

$$\frac{\langle \mathbf{V}\tilde{\mathbf{y}}_e - \mathbf{y}_e, \mathbf{F}(\mathbf{V}\tilde{\mathbf{y}}_e) - \mathbf{F}(\mathbf{y}_e)\rangle}{\|\mathbf{V}\tilde{\mathbf{y}}_e - \mathbf{y}_e\|^2} \le M[\mathbf{F}].$$

By using $\mathbf{F}(\mathbf{y}_e) = 0$ and $\mathbf{p} = \mathbf{F}(\mathbf{V}\tilde{\mathbf{y}}_e)$, we have

$$\frac{\langle \mathbf{V}\tilde{\mathbf{y}}_e - \mathbf{y}_e, \mathbf{p}\rangle}{\|\mathbf{V}\tilde{\mathbf{y}}_e - \mathbf{y}_e\|^2} \le M[\mathbf{F}].$$

Since $M[\mathbf{F}] < 0$, we get

$$\begin{aligned}
\|\mathbf{V}\tilde{\mathbf{y}}_e - \mathbf{y}_e\|^2 &\le \frac{1}{M[\mathbf{F}]}\langle \mathbf{V}\tilde{\mathbf{y}}_e - \mathbf{y}_e, \mathbf{p}\rangle \\
&\le \left|\frac{1}{M[\mathbf{F}]}\right| \|\mathbf{V}\tilde{\mathbf{y}}_e - \mathbf{y}_e\|\,\|\mathbf{p}\| \\
&= \frac{-\|\mathbf{p}\|}{M[\mathbf{F}]}\|\mathbf{V}\tilde{\mathbf{y}}_e - \mathbf{y}_e\|, \\
\|\mathbf{V}\tilde{\mathbf{y}}_e - \mathbf{y}_e\| &\le \frac{-\|\mathbf{p}\|}{M[\mathbf{F}]}.
\end{aligned}$$

∎

Notice that the bound given in (30) can be used to indicate the accuracy of the approximated equilibrium solution $\mathbf{V}\tilde{\mathbf{y}}_e$ from the reduced system (29), even though the exact value of $\mathbf{y}_e$ is not known. In addition, this bound guarantees the convergence of the approximate equilibrium, i.e., $\mathbf{V}\tilde{\mathbf{y}}_e \to \mathbf{y}_e$, as $\|\mathbf{p}\| \to 0$.

Corollary 1. *Suppose the nonlinear vector field* $\mathbf{F}$ *in the full-order system (20) is infinitesimally contracting. Then the nonlinear vector field of the POD reduced system (21) preserves the exponential stability of (20) and has a unique equilibrium* $\tilde{\mathbf{y}}_e^{POD}$. *If* $\mathbf{y}_e$ *is the unique equilibrium solution of (20), then* $\mathbf{y}_e$ *can be approximated by* $\mathbf{V}\tilde{\mathbf{y}}_e^{POD}$ *with the error bound given in (30).*

Proof. This is a direct result from Lemma 2 and Proposition 1 when $\mathbf{W} = \mathbf{I}$. ∎

Although POD reduced system preserves the contractivity of the original system, it does not truly reduce the computational complexity for nonlinear problems, as mentioned earlier. The matrix $\mathbf{W}$ is therefore introduced in (29) It is desirable to choose $\mathbf{W}$ so that the term $\mathbf{V}^T\mathbf{W}\mathbf{F}(\mathbf{W}^T\mathbf{V}\tilde{\mathbf{y}})$ is close to $\mathbf{V}^T\mathbf{F}(\mathbf{V}\tilde{\mathbf{y}})$. Consider the difference of the nonlinear vector field of the POD reduced system (21) and the proposed general form (29)

$$\begin{aligned}
&\|\mathbf{V}^T\mathbf{F}(\mathbf{V}\tilde{\mathbf{y}}) - \mathbf{V}^T\mathbf{W}\mathbf{F}(\mathbf{W}^T\mathbf{V}\tilde{\mathbf{y}})\| \\
&\quad \le \|\mathbf{V}^T\mathbf{F}(\mathbf{V}\tilde{\mathbf{y}}) - \mathbf{V}^T\mathbf{W}\mathbf{F}(\mathbf{V}\tilde{\mathbf{y}})\| \\
&\qquad + \|\mathbf{V}^T\mathbf{W}\mathbf{F}(\mathbf{V}\tilde{\mathbf{y}}) - \mathbf{V}^T\mathbf{W}\mathbf{F}(\mathbf{W}^T\mathbf{V}\tilde{\mathbf{y}})\| \\
&\quad \le \|\mathbf{V}^T - \mathbf{V}^T\mathbf{W}\|\|\mathbf{F}(\mathbf{V}\tilde{\mathbf{y}})\| \\
&\qquad + \|\mathbf{V}^T\mathbf{W}\|\|\mathbf{F}(\mathbf{V}\tilde{\mathbf{y}}) - \mathbf{F}(\mathbf{W}^T\mathbf{V}\tilde{\mathbf{y}})\| \\
&\qquad + \|\mathbf{V}^T\mathbf{W}\|L_F\|(\mathbf{I} - \mathbf{W}^T)\mathbf{V}\tilde{\mathbf{y}}\|,
\end{aligned}$$

where $L_{\mathbf{F}}$ is the Lipschitz constant of $\mathbf{F}$. The above bound for $\|\mathbf{V}^T\mathbf{F}(\mathbf{V}\tilde{\mathbf{y}}) - \mathbf{V}^T\mathbf{W}\mathbf{f}(\mathbf{W}^T\mathbf{V}\tilde{\mathbf{y}})\|$ implies that the difference of the proposed general form and the POD reduced system will be small when $\mathbf{I} - \mathbf{W}$ is close to be orthogonal to the POD basis matrix $\mathbf{V}$. In particular, if we use $\mathbf{W}$ that makes $\mathbf{V} \perp (\mathbf{I} - \mathbf{W})$, we will have $\mathbf{V} = \mathbf{W}^T\mathbf{V}$ and $\|\mathbf{V}^T\mathbf{F}(\mathbf{V}\tilde{\mathbf{y}}) - \mathbf{V}^T\mathbf{W}\mathbf{F}(\mathbf{W}^T\mathbf{V}\tilde{\mathbf{y}})\| = 0$. However, to reduce the computational complexity in practical implementation, we may not be able to have $\mathbf{W}$ that always makes $\mathbf{V} \perp (\mathbf{I} - \mathbf{W})$.

The concept of DEIM will be used to come up with $\mathbf{W}$. However, this cannot be done directly. While the POD reduced system can be shown to be in the form of the reduced system (29), by setting $\mathbf{W} = \mathbf{I}$, the DEIM reduced system cannot be rearranged in this form. Therefore, based on Lemma 2, the POD reduced system preserves the contractivity of the vector field and it inherits other properties of the original full-order system as stated in Corollary 1, but the POD-DEIM approach does not. The following corollary provides the condition that guarantees the contractivity as well as stability, of the resulting POD-DEIM reduced system.

Corollary 2. *Let* $\mathbf{F}$ *be the nonlinear vector field of the full-order system (20). Suppose* $\mathbf{F}$ *is infinitesimally contracting. The corresponding nonlinear vector field* $\widehat{\mathbf{F}}(\tilde{\mathbf{y}}) := \mathbf{V}^T\mathbb{P}\mathbf{F}(\mathbf{V}\tilde{\mathbf{y}})$ *of the POD-DEIM reduced system (27) is infinitesimally contracting if* $M[\mathbb{P}\mathbf{F}] < 0$, *where* $\mathbb{P} = \mathbf{U}(\mathbf{P}^T\mathbf{U})^{-1}\mathbf{P}^T$ *is an oblique projector from the DEIM approximation and,* $\mathbf{V}$ *and* $\mathbf{U}$ *are matrices with orthonormal columns from the POD bases of linear and nonlinear snapshots, respectively.*

Proof. Consider the POD-DEIM reduced system (27) in the form $\dot{\tilde{\mathbf{y}}} = \widehat{\mathbf{F}}(\tilde{\mathbf{y}})$ where $\widehat{\mathbf{F}}(\tilde{\mathbf{y}}) := \mathbf{V}^T\mathbb{P}\mathbf{F}(\mathbf{V}\tilde{\mathbf{y}})$,

$$\begin{aligned}
M[\widehat{\mathbf{F}}] &= \sup_{\hat{\mathbf{u}} \neq \hat{\mathbf{v}}} \frac{(\hat{\mathbf{u}} - \hat{\mathbf{v}})^T(\mathbf{V}^T\mathbb{P}\mathbf{F}(\mathbf{V}\hat{\mathbf{u}}) - \mathbf{V}^T\mathbb{P}\mathbf{F}(\mathbf{V}\hat{\mathbf{v}}))}{\|\hat{\mathbf{u}} - \hat{\mathbf{v}}\|^2} \\
&= \sup_{\hat{\mathbf{u}} \neq \hat{\mathbf{v}}} \frac{(\mathbf{V}\hat{\mathbf{u}} - \mathbf{V}\hat{\mathbf{v}})^T\mathbb{P}(\mathbf{F}(\mathbf{V}\hat{\mathbf{u}}) - \mathbf{F}(\mathbf{V}\hat{\mathbf{v}}))}{\|\mathbf{V}\hat{\mathbf{u}} - \mathbf{V}\hat{\mathbf{v}}\|^2} \\
&\le M[\mathbb{P}\mathbf{F}].
\end{aligned}$$

That is, $M[\mathbb{P}\mathbf{F}] < 0$ implies $M[\widehat{\mathbf{F}}] < 0$. Note that we have used $\|\mathbf{V}\hat{\mathbf{u}} - \mathbf{V}\hat{\mathbf{v}}\|^2 = \|\hat{\mathbf{u}} - \hat{\mathbf{v}}\|^2$, which follows from the orthonormality of the columns in $\mathbf{V}$. ∎

It is desirable to have a model reduction that can both preserve important properties of the original systems and maintain low complexity in computing the projected nonlinear term. To achieve these two desired conditions, the concept of the POD-DEIM approach will be applied to the general form of the contractivity-preserving reduced system (29).

4.2. Proposed specific form of the contractivity-preserving reduced model.

In order to use the general form of the contractivity-preserving reduced system (29) in practice, the matrix $\mathbf{W} \in \mathbb{R}^{n\times n}$ has to be specified. Recall that the matrix $\mathbf{W}$ is initially introduced to reduce the computational complexity of the reduced system (29), which can be done through the concept of interpolatory approximation as described in the DEIM approach. In particular, consider a more specific form of (29) by setting $\mathbf{W} = \mathbf{H}\mathbf{P}^T$, i.e.,

$$\dot{\tilde{\mathbf{y}}} = \widetilde{\mathbf{F}}(\tilde{\mathbf{y}}), \quad \text{with} \quad \widetilde{\mathbf{F}}(\tilde{\mathbf{y}}) = \mathbf{V}^T\mathbf{H}\mathbf{P}^T\mathbf{F}(\mathbf{P}\mathbf{H}^T\mathbf{V}\tilde{\mathbf{y}}), \quad (31)$$

where $\mathbf{P}$ and $\mathbf{H}$ are $n \times m$ matrices. To reduce the computational complexity, the matrix $\mathbf{P} \in \mathbb{R}^{n\times m}$ will be defined as the one given in the DEIM approximation.

For a given $\tilde{\mathbf{y}}$, the function $\widetilde{\mathbf{F}}(\tilde{\mathbf{y}})$ in (31) is computed as grouped below

$$\widetilde{\mathbf{F}}(\tilde{\mathbf{y}}) = \underbrace{\mathbf{V}^T\mathbf{H}}_{k\times m}\underbrace{\mathbf{P}^T\mathbf{F}(\underbrace{\mathbf{P}\mathbf{H}^T\mathbf{V}\tilde{\mathbf{y}}}_{\mathcal{O}(mk)})}_{m\times 1},$$

i.e., the matrix $\mathbf{V}^T\mathbf{H} \in \mathbb{R}^{k\times m}$ can be precomputed in advance and it can also used for $\mathbf{H}^T\mathbf{V} = (\mathbf{V}^T\mathbf{H})^T$; computing $\mathbf{PH}^T\mathbf{V}\tilde{\mathbf{y}}$ can be done with complexity of order $\mathcal{O}(mk)$ due to the special structure of $\mathbf{P}$; and $\mathbf{P}^T\mathbf{F}(\mathbf{PH}^T\mathbf{V}\tilde{\mathbf{y}})$ can be computed with complexity depending on m, since premultiplying by $\mathbf{P}^T$ is simply equivalent to selecting m rows. We will not pre-multiply by $\mathbf{P}$ or $\mathbf{P}^T$ directly in actual computation.

To specify an appropriate matrix $\mathbf{H}$ in (31), consider the POD reduced system (21) and the proposed form (31) before applying the Galerkin projection, i.e., before premultiplying by $\mathbf{V}^T$, which are, respectively, given by

$$\mathbf{V}\dot{\tilde{\mathbf{y}}} = \mathbf{F}(\mathbf{V}\tilde{\mathbf{y}}), \tag{32}$$

$$\mathbf{V}\dot{\tilde{\mathbf{y}}} = \mathbf{HP}^T\mathbf{F}(\mathbf{PH}^T\mathbf{V}\tilde{\mathbf{y}}). \tag{33}$$

The system (32) can be viewed as the original system (20) with projected solution $\mathbf{y} \approx \mathbf{V}\tilde{\mathbf{y}}$. Notice that when the above systems (32) and (33) are pre-multiplied by $\mathbf{V}^T$ in the Galerkin projection, they will become the reduced systems in (21) and (31), respectively.

To make the proposed form (31) consistent with the original system, it is ideal to have the same right-hand-side vector fields of (32) and (33), i.e.,

$$\mathbf{F}(\mathbf{V}\tilde{\mathbf{y}}) = \mathbf{HP}^T\mathbf{F}(\mathbf{PH}^T\mathbf{V}\tilde{\mathbf{y}}). \tag{34}$$

However, the above constraint (34) may not be feasible to specify an appropriate $\mathbf{H}$ in (31). Therefore, this work only aims to have $\mathbf{F}(\mathbf{V}\tilde{\mathbf{y}})$ and $\mathbf{HP}^T\mathbf{F}(\mathbf{PH}^T\mathbf{V}\tilde{\mathbf{y}})$ in (21) and (31), respectively, equal (or as close as possible) at certain selected components determined by DEIM. That is, $\mathbf{H}$ will be chosen to either satisfy

$$\mathbf{P}^T\mathbf{F}(\mathbf{V}\tilde{\mathbf{y}}) = \mathbf{P}^T\mathbf{HP}^T\mathbf{F}(\mathbf{PH}^T\mathbf{V}\tilde{\mathbf{y}}), \tag{35}$$

or solve

$$\min_{\mathbf{H}\in\mathbb{R}^{n\times m}} \|\mathbf{P}^T\mathbf{F}(\mathbf{V}\tilde{\mathbf{y}}) - \mathbf{P}^T\mathbf{HP}^T\mathbf{F}(\mathbf{PH}^T\mathbf{V}\tilde{\mathbf{y}})\|. \tag{36}$$

In the case of a componentwise function $\mathbf{F}$, i.e., $\mathbf{P}^T\mathbf{F}(\mathbf{y}) = \mathbf{F}(\mathbf{P}^T\mathbf{y})$, the vector field in (32) simply becomes $\mathbf{P}^T\mathbf{F}(\mathbf{V}\tilde{\mathbf{y}}) = \mathbf{F}(\mathbf{P}^T\mathbf{V}\tilde{\mathbf{y}}) \in \mathbb{R}^m$, and the vector field in (33) then becomes $\mathbf{P}^T\mathbf{HP}^T\mathbf{F}(\mathbf{PH}^T\mathbf{V}\tilde{\mathbf{y}}) = \mathbf{P}^T\mathbf{HF}(\mathbf{P}^T\mathbf{PH}^T\mathbf{V}\tilde{\mathbf{y}}) = \mathbf{P}^T\mathbf{HF}(\mathbf{H}^T\mathbf{V}\tilde{\mathbf{y}})$. This means that, for a componentwise function $\mathbf{F}$, the condition (35) is equivalent to

$$\mathbf{F}(\mathbf{P}^T\mathbf{V}\tilde{\mathbf{y}}) = \mathbf{P}^T\mathbf{HF}(\mathbf{H}^T\mathbf{V}\tilde{\mathbf{y}}).$$

A straightforward option for $\mathbf{H}$ is obtained by comparing terms in these vector fields, which implies that $\mathbf{H}$ should satisfy $\mathbf{P}^T\mathbf{H} = \mathbf{I} \in \mathbf{R}^{m\times m}$ and $\mathbf{H}^T\mathbf{V} = \mathbf{P}^T\mathbf{V}$. In this case, an obvious choice is therefore $\mathbf{H} = \mathbf{P}$, which will be used in this work. From Lemma 2, another condition that has to be satisfied is that the rank of $\mathbf{W}^T\mathbf{V} = \mathbf{PP}^T\mathbf{V}$ has to be equal to k, i.e., $\mathrm{rank}(\mathbf{PP}^T\mathbf{V}) = k$. In general, this condition may not be true for any matrix $\mathbf{P}$ from DEIM and any POD basis matrix $\mathbf{V}$. It can be shown that this condition fails when k from POD is less than m in DEIM. Hence, the numerical tests of this work always use $m \geq k$ and ensure that $\mathrm{rank}(\mathbf{PP}^T\mathbf{V}) = k$, before applying this approach.

In summary, the final form of reduced system that both preserves contractivity and maintains the low computational complexity is given by

$$\dot{\tilde{\mathbf{y}}} = \mathbf{V}^T\mathbf{PP}^T\mathbf{F}(\mathbf{PP}^T\mathbf{V}\tilde{\mathbf{y}}) \tag{37}$$

with

$$\tilde{\mathbf{y}}(0) = \mathbf{V}^T\mathbf{y}(0),$$

where $\mathbf{P} \in \mathbb{R}^{n\times m}$ and $\mathbf{V} \in \mathbb{R}^{n\times k}$ are defined as in the POD-DEIM reduced system (27) with $m \geq k$ and $\mathrm{rank}(\mathbf{PP}^T\mathbf{V}) = k$. Note that, other forms of $\mathbf{H}$ are possible and left for future research.

In our practical implementation, the vector field $\mathbf{F}$ will be separated into linear and nonlinear terms to maintain the accuracy as much as possible through the linear part. Normally, $\mathbf{F}$ will be written as the sum of two terms: $\mathbf{F}(\mathbf{y}) = \mathbf{Ay} + \mathbf{f}(\mathbf{y})$, where $\mathbf{A} \in \mathbb{R}^{n\times n}$ and $\mathbf{f}(\mathbf{y}) = \mathbf{F}(\mathbf{y}) - \mathbf{Ay}$. In many applications from discretized PDEs, $\mathbf{A}$ is obtained directly from the linear operator, such as the discrete Laplace operator. The POD approach will be applied to the linear term $\mathbf{Ay}$ and reduced system of the form (37) will be used for the nonlinear term $\mathbf{f}(\cdot)$, i.e.,

$$\dot{\tilde{\mathbf{y}}} = \widetilde{\mathbf{A}}\tilde{\mathbf{y}} + \mathbf{V}^T\mathbf{Wf}(\mathbf{W}^T\mathbf{V}\tilde{\mathbf{y}}), \tag{38}$$

where $\widetilde{\mathbf{A}} = \mathbf{V}^T\mathbf{AV} \in \mathbb{R}^{k\times k}$ can be precomputed in advance and $\mathbf{W} = \mathbf{PP}^T$ with $\mathbf{P}$ constructed from DEIM approximation for $\mathbf{f}(\cdot)$. The results on contractivity earlier in (29) and (31) can still be obtained in the same way, because logarithmic Lipschitz constants are subadditive, i.e., $M[\mathbf{F}] = M[\mathbf{A} + \mathbf{f}] \leq M[\mathbf{A}] + M[\mathbf{f}]$, where $M[\mathbf{A}] = \mu[\mathbf{A}]$ for a constant matrix $\mathbf{A} \in \mathbb{R}^{n\times n}$.

Numerical tests in the next section illustrate the efficiency of the proposed model reduction form given in (38) on a nonlinear reaction-diffusion problem.

5. Numerical results

Consider the nonlinear reaction-diffusion initial boundary value problem

$$\frac{\partial u}{\partial t} = \epsilon\frac{\partial^2 u}{\partial x^2} + u - u^3, \qquad x \in \Omega = [x_0, x_f], \quad t \geq 0, \tag{39}$$

with initial condition $u(x, 0)$, and homogeneous boundary conditions $u(x_0, t) = 0$, and $u(x_f, t) = 0$, for $t \geq 0$.

The numerical tests considered in this section use the finite-difference discretizaion with the number of spatial points $n = 1000$ on the domain $\Omega = [0, 1]$ to obtain a full-order system, which can be written in the form:

$$\frac{\mathrm{d}\mathbf{u}}{\mathrm{d}t} = \mathbf{A}\mathbf{u} + \mathbf{f}(\mathbf{u}),$$

where $\mathbf{A} = (\epsilon/h^2)\text{tridiag}[1 \ \ -2 \ \ 1] \in \mathbb{R}^{n\times n}$ constitutes a symmetric tri-diagonal matrix, $\mathbf{f}(\mathbf{u}) = -\mathbf{u}^3 + \mathbf{u}$ is a componentwise nonlinear function with state variable $\mathbf{u} : [0, T] \to \mathbb{R}^n$, $\mathbf{u}_i = \mathbf{u}_i(t) = u(x_i, t)$, $i = 1, 2, \ldots, n$, and $h = (x_f - x_0)/(n + 1)$ is the spatial stepsize with $n + 1$ spatial subintervals (i.e., there are $n + 2$ grid points including the two points on the boundaries). Note that the logarithmic Lipschitz constant of the linear term $\mathbf{A}$ and the nonlinear term $\mathbf{f}$ are given, respectively, by $M[\mathbf{A}] = -(4\epsilon/h^2)\sin^2(\pi h/2) = -\epsilon\pi^2 + \epsilon h^2\pi^4/12 + \mathcal{O}(h^4)$ and $M[\mathbf{f}] = \sup_{\mathbf{u}\in\mathbb{R}^n} \mu[J_f(\mathbf{u})] = 1$, where the Jacobian of $\mathbf{f}$: $J_f(\mathbf{u}) = \text{diag}(-3u_1^2 + 1, -3u_2^2 + 1, \ldots, -3u_n^2 + 1) \in \mathbb{R}^{n\times n}$. The contractivity of the discretized system can be checked through the logarithmic Lipschitz constant $M[\mathbf{A}+\mathbf{f}] \le M[\mathbf{A}]+M[\mathbf{f}] = -\epsilon\pi^2+\epsilon h^2\pi^4/12+\mathcal{O}(h^4)+1$, by using the subadditivity property. When ϵ and h are chosen so that $-\epsilon\pi^2 + \epsilon h^2\pi^4/12 + \mathcal{O}(h^4) + 1 < 0$, it can be guaranteed that the discretized system is contractive, and so are the POD reduced system of the form (21) and the contractivity-preserving POD-DEIM reduced system of the form (38). Two numerical tests will be considered for these contractivity-preserving reduced systems. The first one is for a fixed value of parameter ϵ. The second one is for various different parameter values ϵ, when only one basis set for each of POD and DEIM approximations is used.

5.1. First numerical test: A fixed parameter value.

This numerical test considers the differential equation given in (39) with initial condition $u(x, 0) = \sin(5\pi x)$ and homogeneous boundary conditions $u(0, t) = 0$, $u(1, t) = 0$, for $t \in [0, 0.2]$ and $\epsilon = 0.15$. Figure 1 compares the solutions obtained from the original full-order system of the form (1) with the solutions from two reduced models: (i) the POD reduced system (21), and (ii) the POD-DEIM reduced system that preserves the contractivity (38). It also shows the singular values of the solution snapshots and nonlinear snapshots. As defined earlier, k and m, respectively, denote the dimensions of basis for POD and for DEIM approximation. The numerical solutions in Fig. 1 seem to be indistinguishable for all full-order system with dimension $n = 1000$, POD reduced system with dimension $k = 20$, and POD-DEIM reduced system with dimension $k = m = 20$.

The absolute error and the CPU time (normalized with the simulation time of the original full-order system) of the POD reduced model (21) and contractivity-preserving POD-DEIM reduced model

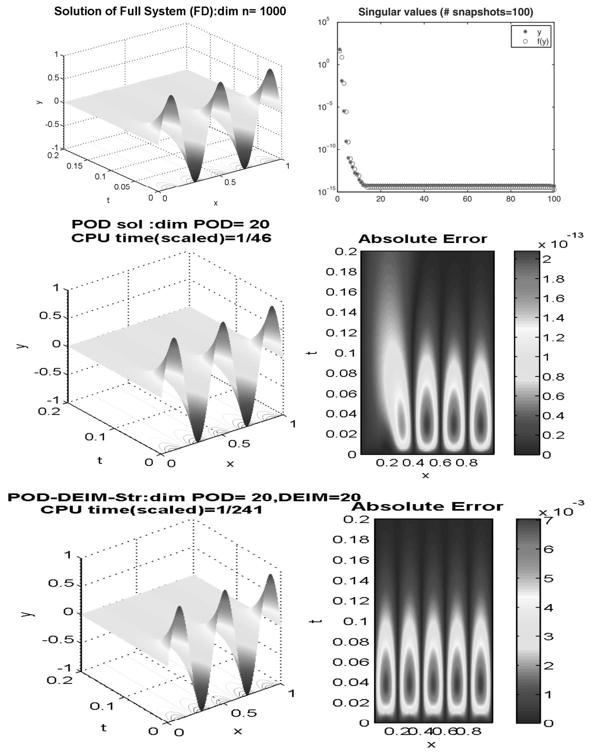

Fig. 1. Numerical test 1: solutions of (39) from the full-order system (1), the POD system (21) with $k = 20$, and the POD-DEIM system that preserves monotonicity (38) with $k = m = 20$.

Table 1. Numerical test 1: run time and relative error of the POD reduced system (top), the POD-DEIM reduced system with monotonicity preserved (bottom). Each run time is normalized with the CPU time of the original full-order system (dimension $n = 1000$).

POD basis (k)	Relative error	Runtime (scaled)
Full: $n = 1000$	–	1
$k = 5$	2.4601e-06	1/56
$k = 10$	3.7046e-08	1/51
$k = 20$	1.7271e-13	1/46
$k = 30$	1.0819e-13	1/39
POD (k) DEIM (m)	**Relative error**	**Runtime (scaled)**
Full: $n = 1000$	–	1
$k = m = 5$	4.1170e-02	1/360
$k = m = 10$	6.2412e-03	1/301
$k = m = 20$	4.1483e-03	1/241
$k = m = 30$	5.4461e-04	1/210

(38) are given in Table 1. Notice that, although the POD approach gives more accurate approximations, the proposed model can accurately approximate the solution with much less simulation time, e.g., the contractivity-preserving POD-DEIM reduced system with $k = 20$, $m = 20$ has CPU time reduced to approximately $1/241$ of the simulation time used for the original system, while CPU time for the POD reduced system with $k = 20$ only reduced to approximately $1/46$ of the time used in the original system. The next numerical test extends these results to various parameter values.

5.2. Second numerical test: Varying parameter values. This section considers an application for the same nonlinear reaction-diffusion equation given in (39) with various values of ϵ. The initial condition is $u(x, 0) = \sin(5\pi x)$, the homogeneous boundary conditions are $u(0, t) = 0, u(1, t) = 0$, for $t \in [0, 1]$. The finite difference discretization is used with the number of spatial points $n = 1000$ on $[0, 1]$ and the number of time steps is $n_t = 100$ on $[0, 1]$.

The POD basis sets used in the Galerkin projection and used for the DEIM nonlinear approximation are constructed from the solution snapshots shown in Fig. 2, which are corresponding to two parameter values $\epsilon = 0.01$ and $\epsilon = 0.1$. The last plot in Fig. 2 shows the singular values corresponding to the POD basis of the solution snapshots from the full-order systems with these two parameter values. The solutions corresponding to different parameter values ϵ in the interval $[0.01, 0.1]$ will be considered here. The plots in Fig. 3 demonstrate, respectively, the solutions from the full-order system, the POD reduced system, and the contractivity-preserving POD-DEIM reduced system with parameter values $\epsilon = 0.015, 0.025$. The solutions from these reduced system in Fig. 3 are shown to accurately capture the dynamics of the original systems although the projection basis sets are constructed from snapshots with different parameter values. Table 2 shows the average relative error and the average runtime of the POD reduced system and the contractivity-preserving POD-DEIM reduced system for parameter $\epsilon = 0.015, 0.020, 0.025, 0.05, 0.07$, which are between the parameter values $0.001, 0.01$ used for constructing POD basis. As in the previous numerical test, each runtime is normalized with the average CPU times of the original full-order systems. Notice that, from Table 2, although the POD approach has higher accuracy, the contractivity-preserving POD-DEIM approach requires an approximately 10 times shorter simulation time than the POD approach.

6. Conclusion

This work proposes a general form of projection-based nonlinear reduced-order modeling that preserves the contractivity property of the original systems, which can be used for guaranteeing the existence, uniqueness of the solution, and stability of the dynamical system. A specific formulation is also proposed by modifying POD and DEIM approaches to satisfy some form of structure. Other specific forms are still possible and left for future research. The numerical tests on the nonlinear reaction diffusion problem demonstrate that, while preserving the contractive property of the original system, the proposed model can accurately approximate the solutions with a much shorter simulation time than both the original model and the standard POD reduced model.

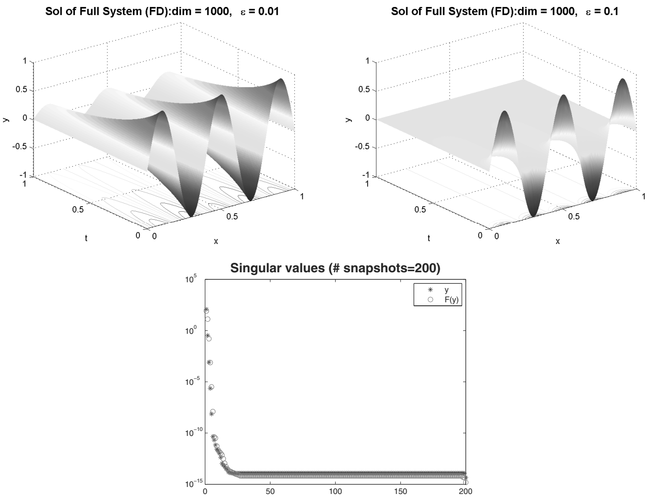

Fig. 2. Numerical test 2: solution snapshots of (39) with dimension $n = 1000$, using $\epsilon = 0.01$ and $\epsilon = 0.1$, and the corresponding singular values.

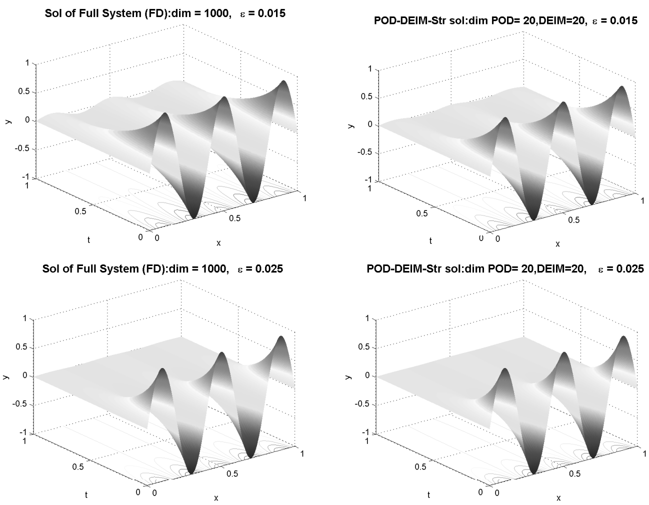

Fig. 3. Numerical test 2: solutions of (39) from the full-order system ($n = 1000$) and the contractivity-preserving POD-DEIM reduced system ($k = m = 20$) with parameter values $\epsilon = 0.015$ and 0.025.

Table 2. Numerical test 2: average relative error and average run time of the POD reduced system (top), the contractivity-preserving POD-DEIM reduced system (bottom) for parameter $\epsilon = 0.015, 0.020, 0.025, 0.05, 0.07 \in [0.001, 0.01]$. Each run time is normalized with the CPU time of the original full-order system (dimension $n = 1000$).

POD basis (k)	Avg. rel. error	Avg. runtime (scaled)
Full: $n = 1000$	–	1
$k = 5$	2.2602e-06	0.02177
$k = 10$	1.4511e-07	0.03225
$k = 20$	5.7141e-11	0.04785
$k = 30$	3.8271e-13	0.05203
$k = 50$	1.1879e-13	0.07952

POD (k) DEIM (m)	Avg. rel. error	Avg. runtime (scaled)
Full: $n = 1000$	–	1
$k = m = 5$	8.1730e-02	0.00220
$k = m = 10$	4.1270e-02	0.00288
$k = m = 20$	1.2412e-03	0.00489
$k = m = 30$	6.4383e-04	0.00505
$k = m = 50$	3.4561e-04	0.00703

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